Matthew R. Watkins

home page

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number theory and physics archive

<< an ever-expanding web-resource documenting the curious, emerging interface between these subjects >>

inexplicable secrets of creation

<< a more speculative, non-technical set of pages largely inspired by the contents of the above site >>

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personal information

I was born in London on 11th June, 1970.

Currently I hold an honorary research position at Exeter University's School of Engineering, Computing and Mathematics.

I completed a Ph.D. in mathematics at the University of Kent at Canterbury in 1994. This essentially concerned the embedding of Riemannian manifolds in spaces of compactly-supported distributions. (And here's my undergraduate dissertation, "A Short Survey of Lens Spaces", which I recently unearthed.)

Immediately thereafter, I took a one-year Royal Society European exchange fellowship at Rijksuniversiteit Gent in Belgium. At the conclusion of this, I had published a few minor papers, but had also gradually become disillusioned with academia (particularly with what I saw as the over-specialisation of modern scientific research). Consequently, I left mathematics to pursue other interests.

The following five years were spent travelling, playing a seven-stringed Turkish instrument (the *saz*), composing, performing and recording music in various contexts, planting trees, investigating a bizarre theory of time and consciousness, setting up an online parapsychology research project, considering possible relationships between Clifford algebras and the ancient Chinese *I Ching* oracle, and writing a little reference book, among other things.

In the Autumn of 1995, whilst staying in the Spanish walled city of Avila, I suddenly became fascinated by (what little I knew about) the distribution of prime numbers. As my earlier mathematical research had not involved number theory, I had much to learn. However, I became convinced by an overwhelming intuition that there was some fundamental link between (i) the distribution of primes and (ii) the ubiquitous Gaussian (probability) distribution.

I sought to learn more, in order to investigate this possibility. Had I been more thorough in my study of the matter, I would have noticed that P. Erdös and M. Kac discovered such a link back in 1940! However, there is some consolation in the fact that the result in question is far from transparent. In fact the deep reasons for this result seem to have mystified even the great Erdös - Carl Pomerance (Erdös number = 1) has suggested that he might (referring to Einstein's famous proclamation) have said something like this:

"God may not play dice with the Universe, but there's something strange going on with the prime numbers!"

(This quote has been misattributed to Erdös himself for many years now.)

In the winter of 1998, having been occasionally contemplating the mystery of the prime distribution for over three years, I was struck by an even more overwhelming intuition about its ultimate nature. My wish to get to the truth of this matter led me back into serious mathematical study, and ultimately to the creation of the number theory and physics archive website.

As my position at Exeter University is honorary, this project is entirely self-funded. In the unlikely instance that you know of any sources of funding for which I might be eligible in order to continue this work, please contact me. Thank you.

Here are a few things I find rather interesting at the moment:

- The fiction of Bruno Schulz
- D. Radin, "Time-reversed human experience: Experimental evidence and implications"
- Vladimir Z. Nuri, "Fractional Reserve Banking as Economic Parasitism: A Scientific, Mathematical & Historical Expose, Critique, and Manifesto" (serious economists who might take issue with this, check second Q&A on p.57)
- The 'Downing Street Memo(s)' - UK and US citizens: please consider writing to your elected representatives to ask what they are doing to have these war criminals removed from office and appropriately dealt with
- *The Great Conjunction* - a mysterious pamphlet jointly authored by the elusive London Psychogeographical Association and the Archaeogeodetic Association, published by Unpopular Books in 1992.
- excerpts from astronomer/philosopher Fred Hoyle's novel *October the First is Too Late*, giving some insight into his deeper thoughts on time and consciousness (which closely mirror my own)

- some notes I compiled a while ago on the etymology of the words for 'left' and 'right' in several languages
- excerpts from an extraordinary 1977 essay by P.K. Dick
- the introduction from the book *Phenomena* by John Michell and Bob Rickard
- John Michell on his discovery regarding the Temple and Old City of Jerusalem
- The Garden of Forking Paths - a site dedicated to J.L. Borges, perhaps my favourite writer
- Lou Kauffman, "Virtual Logic - Formal Arithmetic": an intriguing, poetic look at the number system via G. Spencer Brown's "Laws of Form", informed by mythology and Taoist thought

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[***Critical Strip Explorer***]

[***Communications in Number Theory and Physics***] (new journal)

[a message about the future of this website]

_p_-adic string and brane theories, _etc._

B. Dragovich, ["Some Lagrangians with zeta function nonlocality"] (preprint, 05/2008)

[abstract:] "Some nonlocal and nonpolynomial scalar field models originated from p-adic string theory are considered. Infinite number of spacetime derivatives is governed by the Riemann zeta function through d'Alembertian $\Box$ in its argument. Construction of the corresponding Lagrangians begins with the exact Lagrangian for effective field of p-adic tachyon string, which is generalized replacing p by arbitrary natural number n and then taken a sum of over all n. Some basic classical field properties of these scalar fields are obtained. In particular, some trivial solutions of the equations of motion and their tachyon spectra are presented. Field theory with Riemann zeta function nonlocality is also interesting in its own right."

B. Dragovich, ["Zeta nonlocal scalar fields"] (preprint, 04/2008)

[abstract:] "We consider some nonlocal and nonpolynomial scalar field models originated from p-adic string theory. Infinite number of spacetime derivatives is determined by the operator valued Riemann zeta function through d'Alembertian $\Box$ in its argument. Construction of the corresponding Lagrangians L starts with the exact Lagrangian $\mathcal{L}_p$ for effective field of p-adic tachyon string, which is generalized replacing p by arbitrary natural number n and then taken a sum of $\mathcal{L}_n$ over all n. The corresponding new objects we call zeta scalar strings. Some basic classical field properties of these fields are obtained and presented in this paper. In particular, some solutions of the equations of motion and their tachyon spectra are studied. Field theory with Riemann zeta function dynamics is interesting in its own right as well."

B. Dragovich, ["Zeta strings"] (preprint 03/2007)

[abstract:] "We introduce nonlinear scalar field models for open and open-closed strings with spacetime derivatives encoded in the operator valued Riemann zeta function. The corresponding two Lagrangians are derived in an adelic approach starting from the exact Lagrangians for effective fields of p-adic tachyon strings. As a result tachyons are absent in these models. These new strings we propose to call zeta strings. Some basic classical properties of the zeta strings are obtained and presented in this paper."

B. Dragovich, "On adelic strings" (preprint 05/00)

[abstract:] "New approach to *p*-adic and adelic strings, which takes into account that not only world sheet but also Minkowski space-time and string momenta can be *p*-adic and adelic, is formulated. *p*-Adic and adelic string amplitudes are considered within Feynman's path integral formalism. The adelic Veneziano amplitude is calculated. Some discreteness of string momenta is obtained. Also, adelic coupling constant is equal to unity."

B. Dragovich, "Adelic strings and noncommutativity"

"We consider adelic approach to strings and spatial noncommutativity. Path integral method to string amplitudes is emphasized. Uncertainties in spatial measurements in quantum gravity are related to noncommutativity between coordinates. *p*-Adic and adelic Moyal products are introduced. In particular, *p*-adic and adelic counterparts of some real noncommutative scalar solitons are constructed."

V.S. Vladimirov and Ya.I. Volovich, "On the nonlinear dynamical equation in the *p*-adic string theory" (preprint 06/03)

[abstract:]"In this work nonlinear pseudo-differential equations with the infinite number of derivatives are studied. These equations form a new class of equations which initially appeared in *p*-adic string theory. These equations are of much interest in mathematical physics and its applications in particular in string theory and cosmology. In the present work a systematical mathematical investigation of the properties of these equations is performed. The main theorem of uniqueness in some algebra of tempered distributions is proved. Boundary problems for bounded solutions are studied, the existence of a space-homogenous solution for odd *p* is proved. For even *p* it is proved that there is no continuous solutions and it is pointed to the possibility of existence of discontinuous solutions. Multidimensional equation is also considered and its soliton and *q*-brane solutions are discussed."

I.Ya. Aref'eva, M.G. Ivanov and I.V. Volovich, "Non-extremal intersecting *p*-branes in various dimensions", *Phys. Lett. B* **406** (1997) 44-48

[abstract:] "Non-extremal intersecting *p*-brane solutions of gravity coupled with several antisymmetric fields and dilatons in various space-time dimensions are constructed. The construction uses the same algebraic method of finding solutions as in the extremal case and a modified "no-force" conditions. We justify the "deformation" prescription. It is shown that the non-extremal intersecting *p*-brane solutions satisfy harmonic superposition rule and the intersections of non-extremal *p*-branes are specified by the same characteristic equations for the incidence matrices as for the extremal p-branes. We show that S-duality holds for non-extremal *p*-brane solutions. Generalized T-duality takes place under additional restrictions to the parameters of the theory which are the same as in the extremal case."

I.Ya.Arefeva, K.S.Viswanathan, A.I.Volovich and I.V.Volovich, "Composite *p*-branes in various dimensions", *Nucl. Phys. Proc. Suppl.* **56B** (1997) 52-60

[abstract:] "We review an algebraic method of finding the composite p-brane solutions for a generic Lagrangian, in arbitrary spacetime dimension, describing an interaction of a graviton, a dilaton and one or two antisymmetric tensors. We set the Fock-De Donder harmonic gauge for the metric and the "no-force" condition for the matter fields. Then equations for the antisymmetric field are reduced to the Laplace equation and the equation of motion for the dilaton and the Einstein equations for the metric are reduced to an algebraic equation. Solutions composed of *n* constituent *p*-branes with n independent harmonic functions are given. The form of the solutions demonstrates the harmonic functions superposition rule in diverse dimensions. Relations with known solutions in $D = 10$ and $D = 11$ dimensions are discussed."

I.Ya. Aref'eva, K.S. Viswanathan and I.V. Volovich, "*p*-Brane solutions in diverse dimensions", *Phys.Rev.* **D55** (1997) 4748-4755

[abstract:] "A generic Lagrangian, in arbitrary spacetime dimension, describing the interaction of a graviton, a dilaton and two antisymmetric tensors is considered. An isotropic *p*-brane solution consisting of three blocks and depending on four parameters in the Lagrangian and two arbitrary harmonic functions is obtained. For specific values of parameters in the Lagrangian the solution may be identified with previously known superstring solutions."

I.Ya.Arefeva, K.S.Viswanathan, A.I.Volovich and I.V.Volovich, "Composite *p*-branes in diverse dimensions", *Class. Quant. Grav.* **14** (1997) 2991-3000

[abstract:] "We use a simple algebraic method to find a special class of composite *p*-brane solutions of higher dimensional gravity coupled with matter. These solutions are composed of *n* constituent *p*-branes corresponding n independent harmonic functions. A simple algebraic criteria of existence of such solutions is presented. Relations with $D = 11, 10$ known solutions are discussed."

I.V. Volovich, "*p*-Adic string", *Classical Quantum Gravity* **4** (1987) 83-87

I.V. Volovich, "From *p*-adic strings to étale strings", *Proc. Steklov Inst. Math.* **203** (1995) no. 3, 37–42.

A. Volovich, "Three-block p-branes in various dimensions", *Nucl. Phys.* **B492** (1997) 235-248

[abstract:] "It is shown that a Lagrangian, describing the interaction of the gravitation field with the dilaton and the antisymmetric tensor in arbitrary dimension spacetime, admits an isotropic *p*-brane solution consisting of three blocks. Relations with known *p*-brane solutions are discussed. In particular, in ten-dimensional spacetime the three-block *p*-brane solution is reduced to the known solution, which recently has been used in the D-brane derivation of the black hole entropy."

C. Castro, "Hints of a new relativity principle from *p*-branes quantum mechanics", *Journal of Chaos, Solitons and Fractals* **11** (2000) 1721

C. Castro and A. Granik, "*p*-loops harmonic oscillators in **C**-spaces and the explicit derivation of the black hole entropy"

C. Castro and J. Mahecha, "Comments on the Riemann conjecture and index theory on Cantorian fractal spacetime"

C. Castro (Perelman), "*p*-Adic stochastic dynamics, supersymmetry and the Riemann conjecture"

"Supersymmetry, *p*-adic stochastic dynamics, Brownian motion, Fokker-Planck equation, Langevin equation, prime number random distribution, random matrices, *p*-adic fractal strings, the adelic condition, etc...are all deeply interconnected in this paper."

C. Castro, "Fractal strings as the basis of Cantorian-Fractal spacetime and the fine structure constant"

[abstract:] "Beginning with the most general fractal strings/sprays construction recently expounded in the book by Lapidus and Frankenhuysen, it is shown how the complexified extension of El Naschie's Cantorian-Fractal spacetime model belongs to a very special class of families of fractal strings/sprays whose scaling ratios are given by suitable *p*inary (*p*inary, *p* prime) powers of the Golden Mean. We then proceed to show why the logarithmic periodicity laws in Nature are direct physical consequences of the complex dimensions associated with these fractal strings/sprays. We proceed with a discussion on quasi-crystals with *p*-adic internal symmetries, von Neumann's Continuous Geometry, the role of wild topology in fractal strings/sprays, the Banach-Tarski paradox, tesselations of the hyperbolic plane, quark confinement and the Mersenne-prime hierarchy of bit-string physics in determining the fundamental physical constants in Nature."

P.H. Frampton and Y. Okada, "*p*-Adic string *N*-point function", *Phys. Rev. Lett. B* **60** (1988) 484-486

J. Minahan, "Mode interactions of the tachyon condensate in *p*-adic string theory"

"We study the fluctuation modes for lump solutions of the tachyon effective potential in *p*-adic open string theory. We find a discrete spectrum with equally spaced mass squared levels. We also find that the interactions derived from this field theory are consistent with *p*-adic string amplitudes for excited string states."

A. Sen, "Tachyon condensation and brane descent relations in *p*-adic string theory"

"It has been conjectured that an extremum of the tachyon potential of a bosonic D-brane represents the vacuum without any D-brane, and that various tachyonic lump solutions represent D-branes of lower dimension. We show that the tree level effective action of *p*-adic string theory, the expression for which is known exactly, provides an explicit realisation of these conjectures."

H. Furusho, "*p*-adic multiple zeta values I - *p*-adic multiple polylogarithms and the *p*-adic KZ equation"

[abstract:] "Our main aim in this paper is to give a foundation of the theory of *p*-adic multiple zeta values. We introduce (one variable) *p*-adic multiple polylogarithms by Coleman's *p*-adic iterated integration theory. We define *p*-adic multiple zeta values to be special values of *p*-adic multiple polylogarithms. We consider the *p*-adic KZ equation and introduce the *p*-adic Drinfel'd associator by using certain two fundamental solutions of the *p*-adic KZ equation. We show that our *p*-adic multiple polylogarithms appear on coefficients of a certain fundamental solution of the *p*-adic KZ equation and our *p*-adic multiple zeta values appear on coefficients of the *p*-adic Drinfel'd associator. We show various properties of *p*-adic multiple zeta values, which are sometimes analogous to the complex case and are sometimes peculiar to the *p*-adic case, via the *p*-adic KZ equation."

M. Nardelli, "On the possible mathematical connections between the Hartle-Hawking no boundary proposal concerning the Randall-Sundrum cosmological scenario, Hartle-Hawking wave-function in the mini-superspace of physical superstring theory, *p*-adic Hartle-Hawkind wave function and some sectors of number theory" (preprint, 2007)

M. Nardelli, "On the possible mathematical connections concerning the relation between three-dimensional gravity related to Chern-Simons gauge theory, *p*-adic Hartle-Hawking wave function, Ramanujan's modular functions and some equations describign the Riemann zeta-function" (preprint, 2007)

M. Nardelli, "On the link between the structure of A-branes observed in homological mirror symmetry and the classical theory of automorphic forms. Mathematical connections with the modular elliptic curves, p-adic and adelic numbers and p-adic and adelic strings" (preprint 03/2008)

[abstract:] "This paper is a review of some interesting results that has been obtained in the study of the categories of A-branes on the dual Hitchin fibers and some interesting phenomena associated with the endoscopy in the geometric Langlands correspondence of various authoritative theoretical physicists and mathematicians."

M. Nardelli, "On some mathematical connections concerning the three-dimensional pure quantum gravity with negative cosmological constant, the Selberg zeta-function, the ten-dimensional anomaly cancellations, the vanishing of cosmological constant, and some sectors of string theory and number theory" (preprint 06/2008)

[abstract:] "This paper is a review of some interesting results that has been obtained in the study of the quantum gravity partition functions in three-dimensions, in the Selberg zeta function, in the vanishing of cosmological constant and in the ten-dimensional anomaly cancellations. In the Section 1, we have described some equations concerning the pure three-dimensional quantum gravity with a negative cosmological constant and the pure three-dimensional supergravity partition functions. In the Section 2, we have described some equations concerning the Selberg super-trace formula for Super-Riemann surfaces, some analytic properties of Selberg super zeta-functions and multiloop contributions for the fermionic strings. In the Section 3, we have described some equations concerning the ten-dimensional anomaly cancellations and the vanishing of cosmological constant. In the Section 4, we have described some equations concerning p-adic strings, p-adic and adelic zeta functions and zeta strings. In conclusion, in the Section 5, we have described the possible and very interesting mathematical connections obtained between some equations regarding the various sections and some sectors of number t heory (Riemann zeta functions, Ramanujan modular equations, etc…) and some interesting mathematical applications concerning the Selberg super-zeta functions and some equations regarding the Section 1."

Selberg trace formula and zeta functions

"An important advance was made...in Selberg's paper given at the International Colloquium on Zeta Functions in Bombay in 1956... Selberg discovered that the so called "Poisson summation formula" of classical Fourier analysis had a noncommutative generalization that could be applied to obtain an array of important identities in number theory and the theory of automorphic functions. It is now referred to as the *Selberg trace formula*."

G. Mackey, from *Unitary Group Representations in Physics, Probability,and Number Theory* (Benjamin/Cummings, 1978) p.324 [continued discussion]

"It is easy to understand why Selberg studied trace formulas so intensively: they bear a very striking resemblance to the so-called explicit formulas of prime number theory. Briefly stated, one has:

$$\sum_{\gamma} h(\gamma) = h(\frac{i}{2})+h(-\frac{i}{2})-g(0)\log\pi+\frac{1}{2\pi}\int_{-\infty}^{+\infty} h(r)\frac{\Gamma'}{\Gamma}(\frac{1}{4}+\frac{1}{2}ir)dr-2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n)$$

[*This is the rational (simplest) case of A. Weil's 1952 generalisation of the (number theoretical) explicit formula of Riemann and von Mangoldt. Here h is a complex-valued function of a real variable which satisfies certain conditions, and g is an integral transform of h. Further notes on this formula can be found here. It is very similar in form to a particular case of Selberg's trace formula.*]

where the nontrivial zeros of the Riemann zeta function are denoted by

$$\frac{1}{2}+i\gamma \ (\gamma\in\mathbb{C}) \ "$$

D. Hejhal, *The Selberg Trace Formula for PSL*(2,**R**) - *Volume I*, p. 35

"Selberg noticed this similarity...and was quickly led to a deeper study of trace formulas. Among other things, Selberg found that there is a zeta function which corresponds to [his trace formula] in the same way that [the Riemann zeta function] corresponds to [the Riemann-Weil explicit formula]. This zeta function is nowadays referred to as the *Selberg zeta function*; it is usually denoted by *Z*(*s*)."

D. Hejhal, "The Selberg trace formula and the Riemann zeta function", *Duke Mathematics Journal* **43** (1976) p.459

"Motivated by the mysterious resemblance of the Selberg trace formula to this explicit formula, Selberg introduced [the Selberg zeta function:]

$$Z(s) = \prod_{\{\gamma\}\,\text{primitive}} \prod_{k=0}^{\infty} (1 - N_0(\gamma)^{-s-k}).$$

Its zeros are *s* with *s*(1-*s*) an eigenvalue of [the Laplace-Beltrami operator], so *Z*(*s*) = 0 implies *s* = 1/2 + *it*, *t* real or *s* in [0,1]."

D. Bump, "Spectral Theory of SL(2,**R**)" (Jerusalem 2001 lecture notes)

"Besides a series of 'trivial' roots, all the roots of Selberg's zeta function fall on the line 1/2 + *i*R: in fact they are precisely the numbers 1/2 + *ik* for which $-(1/4 + k)^2$ is an eigenvalue of [the Laplacian]. This is the 'Riemann conjecture' in the present case."

H.P. McKean, "Selberg's trace formula as applied to a compact Riemannian surface", *Communications in Pure and Applied Mathematics* **25** (1972) 225-246.

"The Selberg trace formula was originally introduced as an arithmetical relation, being a noncommutative generalisation of the Poisson summation formula, and it is used as such in number theory and harmonic analysis. In the present context we view it as an *identity* relating *dynamical quantitities*, the quantal spectrum of the Laplace-Beltrami operator and the classical 'length spectrum' of periodic geodesics, both being defined on a configuration space of the form *D*/*G*."

A. Voros and N.L. Balasz, "Chaos on the pseudosphere", *Physics Reports* **143** no. 3, p. 169.

"On a compact (i.e. closed and bounded) two-dimensional surface of negative Gaussian curvature the classical motion [of a point mass] takes place on the geodesics, and it is as chaotic and nonintegrable as possible (being Bernoullian). On this surface there exists also a well-defined quantum dynamics, where the Laplace-Beltrami operator (the invariant Laplacian) acts as the Hamiltonian in the Schrödinger equation. A limiting procedure, exactly parallel to the semiclassical tradition in ordinary quantum mechanics, takes the quantum theory into the classical one when the energy E becomes large, $E^{-1/2}$ playing the role of Planck's constant...(We note that the general "inverse problem" of quantising classical mechanics on a curved space, which presents its own difficulties of another order, does not arise here.) If in addition the curvature is *constant*, this semiclassical transition is even understood in a certain sense, exemplified by the Selberg trace formula [5]. This formula, which was motivated by Riemann's zeta function, relates in an exact way the spectrum of the quantal motion on compact surfaces of negative curvature to the classical motion. The so-resulting mathematical literature [6] has deep connections with manifold theory, automorphic functions, number theory, etc.; however, it does not address itself explicitly to questions which are of crucial interest to the physicist, i.e., the detailed properties of the discrete energy spectrum and associated eigenfunctions, and their relation to questions of quantum ergodicity and quantal chaos.

Physicists were not much concerned by the purely classical aspects of this model. For them ergodicity and mixing were the consequence of forces of interaction between particles, and not of physically nonrealisable constraint forces. However the situation becomes different if we view this as an exactly soluble classical model to be quantised, in order to study quantum-mechanically the manifestation of chaoticity. Nevertheless, very little use has been made of this model. In effect, only two pioneering studies have appeared in this direction until now. Gutzwiller [7] has drawn attention to the relation between Selberg's trace formula and the semiclassical expansion of Green's function described by a path integral. He also studied [8] the scattering on a compact surface of constant negative curvature using the work of Lax and Phillips [9]."

A. Voros and N.L. Balasz, "Chaos on the pseudosphere", *Physics Reports* **143** no. 3, p. 112.

[5] A. Selberg, "Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series", *Journal of the Indian Mathematical Society* **20** (1956) 47-87.
[6] Hejhal, D., *The Selberg Trace Formula for PSL(2,**R**), Volume I* (Springer Lecture Notes **548**, 1976)
[7] M.C. Gutzwiller, *Physical Review Letters* **45** (1980) 150-153.
[8] M.C. Gutzwiller, *Physica* D **7** (1983) 341-355.
[9] P.D. Lax and R.S. Phillips, *Scattering Theory for Automorphic Functions* (Princeton University Press, 1976)

"The classical periodic orbits are a crucial stepping stone in the understanding of quantum mechanics, in particular when then classical system is chaotic. This situation is very satisfying when one thinks of Poincaré who emphasized the importance of periodic orbits in classical mechanics, but could not have had any idea of what they could mean for quantum mechanics. The set of energy levels and the set of periodic orbits are complementary to each other since they are essentially related through a Fourier transform. Such a relation had been found earlier by the mathematicians in the study of the Laplacian operator on Riemannian surfaces with constant negative curvature. This led to Selberg's trace formula in 1956 which has exactly the same form, but happens to be exact. The mathematical proof, however, is based on the high degree of symmetry of these surfaces which can be compared to the sphere, although the negative curvature allows for many more different shapes."

M.C. Gutzwiller, "Chaos in Quantum Mechanics" (1998 lecture notes)

extensive notes from Gutzwiller on the Selberg Trace Formula and quantum chaos

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Unfortunately there are few good references on the Selberg trace formula. Peter Perry suggested the following:

H.P. McKean, "Selberg's trace formula as applied to a compact Riemannian surface", *Communications in Pure and Applied Mathematics* **25** (1972) 225-246. See also the relevant erratum in *Comm. Pure Appl. Math.* **27** (1974) p.134.

"This will give a somewhat breezy but in principal complete derivation of Selberg's trace formula for a compact Riemann surface - where the Laplacian has only eigenvalues and no continuous spectrum. There are generalizations to cases where the underlying manifold is non-compact and the Laplacian has continuous spectrum, but these are much more involved analytically.

A useful analogy is Poisson's summation formula for a torus, thought of as a relation between the eigenvalues of the Laplacian (*e.g.* $4(pi)^2 n^2$ on the circle viewed as [0,1)) and the lengths of closed geodesics (*n* for integers *n*). For a Schwarz class test function the Poisson summation formula says that

$$\sum_n f(n) = \sum_n \hat{f}(2\pi n)$$

and so relates the test function evaluated at lengths on the left to its Fourier transform evaluation at square roots of eigenvalues on the right).

Selberg's trace formula is the spiritual ancestor of the celebrated Duistermaat-Guillemin trace formula. See:

J. Duistermaat and V. Guillemin "The spectrum of positive elliptic operators and periodic bicharacteristics", *Invent. Math.* **29** (1975), no. 1, 39-79."

A. Selberg, "Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series", *Journal of the Indian Mathematical Society* **20** (1956) 47-87.

This is the article in which the trace formula first appeared:

[abstract] "In the following lectures we shall give a brief sketch of some representative parts of certain investigations that have been undertaken during the last five years. The center of these investigations is a general relation which can be considered as a generalization of the so-called Poisson summation formula (in one or more dimensions). This relation we here refer to as the "trace-formula.""

What is often referred to in the literature as "the Selberg trace formula" is actually a particular case of the general formula Selberg proves in this paper.

The general formula he develops in section 2 applies to a Riemannian manifold *S* of any dimension, together with a locally compact group of isometries *G* acting on *S*, and a discrete subgroup of *G*.

In section 3, he treats the special case where *S* is the hyperbolic plane. *G* and its subgroup are chosen appropriately so that the formula relates to compact Riemannian surfaces (which are well-known to be expressible as quotients of the hyperbolic plane).

This is the formula as given on p.74 of Selberg's paper:

$$\sum_i h(r_i) = \frac{A(\mathscr{D})\,\nu}{2\pi}\int_{-\infty}^{\infty} r\,\frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}\,h(r)\,dr\; +$$
$$+ \sum_{\{R\}_\Gamma}\sum_{k=1}^{m-1}\frac{\sigma(\chi^k(R))}{M\sin k\pi/m}\int_{-\infty}^{\infty}\frac{e^{-2\pi rk/m}}{1+e^{2-\pi r}}\,h(r)\,dr\; +$$
$$+\, 2\sum_{\{P\}_\Gamma}\sum_{k=1}^{\infty}\frac{\sigma(\chi^k(P))\log N\{P\}}{(N\{P\})^{k/2}-(N\{P\})^{-k/2}}\,g(k\log N\{P\}).$$

The left-hand side is effectively a sum over the Laplacian spectrum (*i.e.* the harmonic frequencies) of the compact Riemannian surface. Both left and right hand sides can be understood as distributions, where *h* is the 'test-function' being acted on (*g* is just an integral transform of *h*).

The second and third terms of the right-hand side are sums over *primitive conjugacy classes* of isometries on the surface. The {*R*} refers to conjugacy classes of elliptic isometries, and {*P*} to conjugacy classes of hyperbolic isometries. Note that Hejhal's 1976 paper (mentioned above) presents a simpler version where there are no elliptic conjugacy classes.

The $N\{P\}$ in the third term are crucial here. They are the *norms* of primitive conjugacy classes of hyperbolic isometries. Roughly speaking the norm measures the factor by which such an isometry 'dilates' the hyperbolic plane (which is the universal covering surface of the compact Riemann surface in question). These norms are closely related to the lengths of primitive geodesics on the surface and, in the setting of (quantum) chaos, the periods of primitive orbits in certain flows.

The resemblance between this formula and the Riemann-Weil explicit formula is such that the $N\{P\}$ correspond to the prime numbers, and the r_i on the left-hand side (which are directly linked to the Laplacian spectrum of the surface) correspond to the nontrivial zeros of the Riemann zeta function. Consequently, the resemblance is a major source of support for the spectral interpretation of the Riemann zeta function. Put very simply, the spectral interpretation argues that "the nontrivial zeros of the Riemann zeta function are eigenvalues in some setting".

The following survey article explains in detail the relationship between the Selberg trace formula and the Riemann zeta function:

D. Hejhal, "The Selberg trace formula and the Riemann zeta function", *Duke Mathematics Journal* **43** (1976) 441-482.

Also highly relevant:

D. Goldfeld, "Explicit Formulae as Trace Formulae", from *Number Theory, Trace Formulas and Discrete Groups* (K.E. Aubert, E. Bombieri and D. Goldfeld, eds.) (Academic, 1989) 281-288

"In his epoch-making paper, Selberg developed a general trace formula for discrete subgroups of GL(2,**R**). The analogies with the explicit formulae of Weil (relating very general sums over primes with corresponding sums over the critical zeroes of the zeta-function) are quite striking and have been the subject of much speculation over the years.

It is the object of this note to show that Weil's explicit formula can in fact be interpreted as a trace formula on a suitable space. The simplest space we have been able to construct for this purpose, at present, is the semidirect product of the ideles of norm one with the adeles, factored by the discrete subgroup **Q*** |X **Q**, the semidirect product of the multiplicative group of rational numbers with the additive group of rational numbers. We will show that for a suitable kernel function on this space, the conjugacy class side of the Selberg trace formula, is precisely the sum over the primes occurring in Weil's explicit formula.

This implies that the sum of the eigenvalues of the self-adjoint integral operator associated to the aforementioned kernel function is precisely the sum over the critical zeroes of the Riemann zeta-function occurring on the other side of Weil's formula. The relation between the eigenvalues of this integral operator and the zeroes of the zeta-function appears quite mysterious at present. What is lacking is a suitable generalization of the Selberg transform in this situation.

Finally, we should point out that our approach leads to various new equivalences to the Riemann Hypothesis, such as certain positivity hypotheses for the integral operators.

Although we have worked over **Q**, for simplicity of exposition, it is not hard to generalize our relults to *L*-functions of arbitrary number fields."

Andy Sinton of UC Berkeley recently informed me (December 2003):

"Jay Jorgenson and Serge Lang are near completion on results that obtain the functional equation of Selberg zeta (or it's logarithmic derivative), they're also working on SL(2,**C**) as a Poisson summation despite Iwaniec's claim that "*the functional equation...resists any decent interpretation as a kind of Poisson summation principle*". Lang is teaching about it at Yale and there should be some lecture notes available sometime in the coming year."

Dan Bump suggests the following:

Iwaniec, H., *Introduction to the Spectral Theory of Automorphic Forms*, 2nd edition, Graduate Studies in Mathematics **53** (AMS, 2002)

Chapter 10 covers the Trace Formula, and on p.154 we find a helpful note about Selberg zeta functions:

"If you will, the Selberg zeta-function satisfies an analogue of the Riemann hypothesis. However, the analogy with the Riemann zeta-function is superficial. First of all, the Selberg zeta function has no natural development into Dirichlet series. Furthermore, the functional equation...resists any decent interpretation as a kind of Poisson summation principle. Nevertheless, modern studies of $Z(s)$ have caused a lot of excitement in mathematical physics (see [Sa1]). At least, one may say that the dream of Hilbert and Pólya of connecting the zeros of a zeta-function with eigenvalues of a self-adjoint operator is a reality in the context of $Z(s)$."

[Sa1] P. Sarnak, "Determinants of Laplacians", *Communications in Mathematical Physics* **110** (1987) 113-120.

Venkov, A., *Spectral Theory of Automorphic Functions and its Applications*, Mathematics and its Applications (Soviet Series), 51 (Kluwer, 1990)

Tamagawa, T., "On Selberg's trace formula", *J. Fac. Sci. Univ. Tokyo* Sect. I **8** (1960) 363-386

Hejhal, D., *The Selberg Trace Formula for PSL(2,**R**), Volume I* (Springer Lecture Notes **548**, 1976)

"The Tamagawa paper is important because it is the origin of the representation theoretic approach to the trace formula."

Bump's 2001 lecture notes "Spectral Theory of SL(2,**R**)" include a proof of the Selberg trace formula.

[PDF] [DVI]

See also:

J. Arthur "The Trace Formula and Hecke Operators", from *Number Theory, Trace Formulas and Discrete Groups* (K.E. Aubert, E. Bombieri and D. Goldfeld, eds.) (Academic, 1989)

"This lecture is intended as a general introduction to the trace formula. We shall describe a formula that is a natural generalization of the Selberg trace formula for compact quotient. Selberg also established trace formulas for noncompact quotients of rank 1, and our formula can be regarded as an analogue for general rank of these. As an application, we shall look at the 'finite case' of the trace formula. We shall describe a finite closed formula for the traces of Hecke operators on certain eigenspaces."

Andrew Bennett's very helpful applets for exploring the hyperbolic plane

A. Juhl's history of generalisations of Selberg zeta functions

D. Hejhal, "The Selberg trace formula for congruence subgroups", *Bulletin of the AMS* **81** (1975) 752-755.

C. Grosche, "Selberg supertrace formula for super Riemann surfaces III: bordered super Riemann surfaces", *Commun. Math. Phys.* **162** (1994) 591-632

"The analytic properties of the Selberg super zeta-functions on bordered super Riemann surfaces are discussed, and super-determinants of Dirac-Laplace operators on bordered super Riemann surfaces are calculated in terms of Selberg super zeta-functions."

U. Bunke and M. Olbrich, "Group cohomology and the singularities of the Selberg zeta function associated to a Kleinian group", *Annals of Mathematics* **149** no.2 (1999) 627-689

[abstract:] "We prove Patterson's conjecture about the singularities of the Selberg zeta function associated to a convex-cocompact, torsion free group acting on a hyperbolic space."

U. Bunke and M. Olbrich, "Gamma cohomology and the Selberg zeta function" (preprint 11/94)

[abstract:] "We propose a new method for studying n- and Γ-cohomology of globalizations of Harish-Chandra modules, where $G=KAN$ is a rank one semisimple Lie group, Γ is a discrete subgroup of G and $n=Lie(N)$. We prove a conjecture of Patterson relating the singularities of Selberg zeta functions with the Γ-

cohomology of principal series representations if Γ is cocompact and torsion free."

P. Perry, "Resonances, zeta functions, and trace formulas for Kleinian groups" (preprint? *Note: this PDF contains an irregularity in that the last first page appears as last page.*)

P. Perry and S. Patterson, "The divisor of Selberg's zeta function for Kleinian groups", *Duke Math. J.* **106** No.2 (2001)

[summary by Peter G. Gilkay:] "Selberg's zeta-function is defined in terms of the length spectrum of a compact or, more generally, a finite volume hyperbolic surface. Its divisor is determined by the eigenvalues and scattering poles of the Laplacian and the Euler characteristic of the surface. The authors note "this relationship may be regarded as one form of Selberg's trace formula for a compact or finite volume surface and as a particularly precise quantization of an ergodic system, namely, the geodesic flow".

The authors extend this relationship to convex cocompact hyperbolic manifolds in all dimensions. They show the divisor of the Selberg zeta-function is determined by the eigenvalues and scattering poles of the Laplacian together with the Euler characteristic of the corresponding compactified manifold with boundary. In even dimensions, this uses work of Bunke and Olbrich.

The contents of the paper are as follows: **1.** Introduction. **2.** Symbolic dynamics and Selberg's zeta-function. **3.** Geometry at infinity (coordinate neighborhoods, envelopes of horospheres, normal flow). **4.** Review of scattering theory (model space, resolvant, eigenfunctions, operator). **5.** The 'logarithmic derivative' of the scattering operator. **6.** The logarithmic derivative of the zeta-function. **7.** Computation of the divisor (spectral term, topological term, residues in odd and even dimensions).
Appendix A: An asymptotic volume formula for convex cocompact hyperbolic manifolds (by C. Epstein).
Appendix B: The scattering operator and zeta-function for a class of cylindrical manifolds."

E. Bogomolny, "Quantum and arithmetical chaos" (preprint 12/03, based on lectures given at Les Houches School "Frontiers in Number Theory, Physics and Geometry", March 2003)

[abstract:] "The lectures are centered around three selected topics of quantum chaos: the Selberg trace formula, the two-point spectral correlation functions of Riemann zeta function zeros, and of the Laplace-Beltrami operator for the modular group. The lectures cover a wide range of quantum chaos applications and can serve as a non-formal introduction to mathematical methods of quantum chaos."

D. Mayer, "The thermodynamic formalism approach to Selberg's zeta function for PSL(2,**Z**)", *Bulletin of the AMS* **25** (1991) 55-60.

"The thermodynamic formalism...leads to a rather explicit representation of the Smale-Ruelle function and hence also of the Selberg function for PSL(2,**Z**) in terms of Fredholm determinants of transfer operators of the map T_G. Finally, combining our results with classical ones for the Selberg function derived from the trace formula suggests also a seemingly new formulation of Riemann's hypothesis on his zeta function in terms of the transfer operators of T_G."

Cheng-Hung Chang and D. Mayer, "The transfer operator to Selberg's zeta function and modular Maass waveforms for PSL(2,**Z**)", from *Emerging Applications of Number Theory* (*Mathematics and its Applications* vol. 109, eds. D. Hejhal, M. Gutzwiller, *et. al.*) (Springer, 1999)

Cheng-Hung Chang and D. Mayer, "Thermodynamic formalism and Selberg's zeta function for modular groups", *Regular and Chaotic Dynamics* **5** no.3 (2000) 281-312

Cheng-Hung Chang and D. Mayer, "Eigenfunctions of the transfer operators and period functions for modular groups", from *Dynamical, Spectral, and Arithmetic Zeta Functions (San Antonio, TX, 1999)*, (Contemporary Mathematics **290**) (AMS, 2001) 1-40

D. Zagier and J. Lewis, "Period functions and the Selberg zeta function of the modular group", from *The Mathematical Beauty of Physics*, *Adv. Series in Math. Physics* **24** (World Scientic, 1997) 83-97

A. Terras and D. Wallace, "Selberg's trace formula on the *k*-regular tree and applications", ,i>Int. J. of Math. and Math. Sci. **8** (2003) 501-526.

J.S. Friedman, "The Selberg trace formula and Selberg zeta-function for cofinite Kleinian groups with finite-dimensional unitary representations" (preprint 10/04)

[abstract:] "For cofinite Kleinian groups, with finite-dimensional unitary representations, we derive the Selberg trace formula. As an application we define the corresponding Selberg zeta-function and compute its divisor, thus generalizing results of Elstrodt, Grunewald and Mennicke to non-trivial unitary representations. We show that the presence of cuspidal elliptic elements sometimes adds ramification point to the zeta function. In fact, if D is the ring of Eisenstein integers, then the Selberg zeta-function of PSL(2,D) contains ramification points and is the sixth-root of a meromorphic function."

J. Fiala and P. Kleban, "Generalized number theoretic spin chain-connections to dynamical systems and expectation values", *J. of Stat. Physics* **121** (2005) 553-577

[abstract:] "We generalize the number theoretic spin chain, a one-dimensional statistical model based on the Farey fractions, by introducing a new parameter $x \geq 0$. This allows us to write recursion relations in the length of the chain. **These relations are closely related to the Lewis three-term equation, which is useful in the study of the Selberg zeta-function.** We then make use of these relations and spin orientation transformations. We find a simple connection with the transfer operator of a model of intermittency in dynamical systems. In addition, we are able to calculate certain spin expectation values explicitly in terms of the free energy or correlation length. Some of these expectation values appear to be directly connected with the mechanism of the phase transition."

P.A. Perry and F.L. Williams, "Selberg zeta function and trace formula for the BTZ black hole" (preprint 02/03)

[abstract:] "A Selberg zeta function is attached to the three-dimensional BTZ black hole, and a trace formula is developed for a general class of test functions. The trace formula differs from those of more standard use in physics in that the black hole has a fundamental domain of infinite hyperbolic volume. Various thermodynamic quantities associated with the black hole are conveniently expressed in terms of the zeta function."

M. Nardelli, "On some mathematical connections concerning the three-dimensional pure quantum gravity with negative cosmological constant, the Selberg zeta-function, the ten-dimensional anomaly cancellations, the vanishing of cosmological constant, and some sectors of string theory and number theory" (preprint 06/2008)

[abstract:] "This paper is a review of some interesting results that has been obtained in the study of the quantum gravity partition functions in three-dimensions, in the Selberg zeta function, in the vanishing of cosmological constant and in the ten-dimensional anomaly cancellations. In the Section 1, we have described some equations concerning the pure three-dimensional quantum gravity with a negative cosmological constant and the pure three-dimensional supergravity partition functions. In the Section 2, we have described some equations concerning the Selberg super-trace formula for Super-Riemann surfaces, some analytic properties of Selberg super zeta-functions and multiloop contributions for the fermionic strings. In the Section 3, we have described some equations concerning the ten-dimensional anomaly cancellations and the vanishing of cosmological constant. In the Section 4, we have described some equations concerning p-adic strings, p-adic and adelic zeta functions and zeta strings. In conclusion, in the Section 5, we have described the possible and very interesting mathematical connections obtained between some equations regarding the various sections and some sectors of number t heory (Riemann zeta functions, Ramanujan modular equations, etc…) and some interesting mathematical applications concerning the Selberg super-zeta functions and some equations regarding the Section 1."

D. Fried, "The zeta functions of Ruelle and Selberg. I.", *Ann. Sci. Ecole Norm. Sup.* (4) **19** (1986), 491-517.

A. Venkov and A. Nikitin, "Selberg trace formula, Ramanujan graphs and some problems of mathematical physics", *Algebra Anal*. **5** (1993) 1-76.

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D. Wallace, "The Selberg trace formula for SL(3,**Z**)\ SL(3,**R**)/ SO(3,**R**)", *Transanctions of the AMS* (new series) **345** (1994), 136.

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string theory, quantum cosmology, *etc.*

E. Elizalde, S. Leseduarte and S. Zerbini, "Mellin transform techniques for zeta-function resummations"

"Making use of inverse Mellin transform techniques for analytical continuation, an elegant proof and an extension of the zeta function regularization theorem is obtained...As an application of the method, the summation of the series which appear in the analytic computation (for different ranges of temperature) of the partition function of the string - basic in order to ascertain if QCD is some limit of a string theory - is performed. "

E. Elizalde, S. Leseduarte and S.D. Odintsov, "Partition functions for the rigid string and membrane at any temperature"

"Exact expressions for the partition functions of the rigid string and membrane at any temperature are obtained in terms of hypergeometric functions. By using zeta function regularization methods, the results are analytically continued and written as asymptotic sums of Riemann-Hurwitz zeta functions, which provide very good numerical approximations with just a few first terms."

A class of zeta functions that extends the class of Epstein's was recently brought to my attention by Prof. E. Elizalde of M.I.T. They are spectral zeta functions associated with a quadratic + linear + constant form in any number of dimensions. Elizalde has developed formulas for them which extend the famous Chowla-Selberg formula.

E. Elizalde, "Explicit zeta functions for bosonic and fermionic fields on a noncommutative toroidal spacetime", *Journal of Physics A* **34** (2001) 3025-3036.

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E. Elizalde, "Zeta functions, formulas and applications", *J. Comp. Appl. Math.* **118** (2000) 125.

G.W. Moore, "Les Houches lectures on strings and arithmetic" (preprint 01/04) [abstract:] "These are lecture notes for two lectures delivered at the Les Houches workshop on Number Theory, Physics, and Geometry, March 2003. They review two examples of interesting interactions between number theory and string compactification, and raise some new questions and issues in the context of those examples. The first example concerns the role of the Rademacher expansion of coefficients of modular forms in the AdS/CFT correspondence. The second example concerns the role of the "attractor mechanism" of supergravity in selecting certain arithmetic Calabi-Yau's as distinguished compactifications."

B. Dragovich, "Some Lagrangians with zeta function nonlocality" (preprint, 05/2008)

[abstract:] "Some nonlocal and nonpolynomial scalar field models originated from p-adic string theory are considered. Infinite number of spacetime derivatives is governed by the Riemann zeta function through d'Alembertian $\Box$ in its argument. Construction of the corresponding Lagrangians begins with the exact Lagrangian for effective field of p-adic tachyon string, which is generalized replacing p by arbitrary natural number n and then taken a sum of over all n. Some basic classical field properties of these scalar fields are obtained. In particular, some trivial solutions of the equations of motion and their tachyon spectra are presented. Field theory with Riemann zeta function nonlocality is also interesting in its own right."

B. Dragovich, "Zeta nonlocal scalar fields" (preprint, 04/2008)

[abstract:] "We consider some nonlocal and nonpolynomial scalar field models originated from p-adic string theory. Infinite number of spacetime derivatives is determined by the operator valued Riemann zeta function through d'Alembertian $\Box$ in its argument. Construction of the corresponding Lagrangians L starts with the exact Lagrangian $\mathcal{L}_p$ for effective field of p-adic tachyon string, which is generalized replacing p by arbitrary natural number n and then taken a sum of $\mathcal{L}_n$ over all n. The corresponding new objects we call zeta scalar strings. Some basic classical field properties of these fields are obtained and presented in this paper. In particular, some solutions of the equations of motion and their tachyon spectra are studied. Field theory with Riemann zeta function dynamics is interesting in its own right as well."

B. Dragovich, "Zeta strings" (preprint 03/2007)

[abstract:] "We introduce nonlinear scalar field models for open and open-closed strings with spacetime derivatives encoded in the operator valued Riemann zeta function. The corresponding two Lagrangians are derived in an adelic approach starting from the exact Lagrangians for effective fields of p-adic tachyon strings. As a result tachyons are absent in these models. These new strings we propose to call zeta strings. Some basic classical properties of the zeta strings are obtained and presented in this paper."

M. McGuigan, "Riemann Hypothesis and short distance fermionic Green's functions" (preprint 04/05) [abstract:] "We show that the Green's function of a two dimensional fermion with a modified dispersion relation and short distance parameter *a* is given by the Lerch zeta function. The Green's function is defined on a cylinder of radius *R* and we show that the condition *R* = *a* yields the Riemann zeta function as a quantum transition amplitude for the fermion. We formulate the Riemann hypothesis physically as a nonzero condition on the transition amplitude between two special states associated with the point of origin and a point half way around the cylinder each of which are fixed points of a Z_2 transformation. By studying partial sums we show that that the transition amplitude formulation is analogous to neutrino mixing in a low dimensional context. We also derive the thermal partition function of the fermionic theory and the thermal divergence at temperature 1/*a*. In an alternative harmonic oscillator formalism we discuss the relation to the fermionic description of two dimensional string theory and matrix models. Finally we derive various representations of the Green's function using energy momentum integrals, point particle path integrals, and string propagators."

M. McGuigan, "Riemann Hypothesis, matrix/gravity correspondence and FZZT brane partition functions" (preprint 08/2007)

[abstract:] "We investigate the physical interpretation of the Riemann zeta function as a FZZT brane partition function associated with a matrix/gravity correspondence. The Hilbert-Polya operator in this interpretation is the master matrix of the large N matrix model. Using a related function $\Xi(z)$ we develop an analogy between this function and the Airy function Ai(z) of the Gaussian matrix model. The analogy gives an intuitive physical reason why the zeros lie on a critical line. Using a Fourier transform of the $\Xi(z)$ function we identify a Kontsevich integrand. Generalizing this integrand to $n \times n$ matrices we develop a Kontsevich matrix model which describes n FZZT branes. The Kontsevich model associated with the $\Xi(z)$ function is given by a superposition of Liouville type matrix models that have been used to describe matrix model instantons."

M. McGuigan, "Riemann Hypothesis and master matrix for FZZT brane partition functions" (preprint 05/2008)

[abstract:] "We continue to investigate the physical interpretation of the Riemann zeta function as a FZZT brane partition function associated with a matrix/gravity correspondence begun in arxiv:0708.0645. We derive the master matrix of the $(2,1)$ minimal and $(3,1)$ minimal matrix model. We use it's characteristic polynomial to understand why the zeros of the FZZT partition function, which is the Airy function, lie on the real axis. We also introduce an iterative procedure that can describe the Riemann Ξ function as a deformed minimal model whose deformation parameters are related to a Konsevich integrand. Finally we discuss the relation of our work to other approaches to the Riemann Ξ function including expansion in terms of Meixner-Pollaczek polynomials and Riemann-Hilbert problems."

H.C. Rosu, "Quantum Hamiltonians and prime numbers", *Modern Physics Letters* A **18** (2003)

[abstract:] "A short review of Schroedinger hamiltonians for which the spectral problem has been related in the literature to the distribution of the prime numbers is presented here. **We**

notice a possible connection between prime numbers and centrifugal inversions in black holes and suggest that this remarkable link could be directly studied within trapped Bose-Einstein condensates. In addition, when referring to the factorizing operators of Pitkanen and Castro and collaborators, we perform a mathematical extension allowing a more standard supersymmetric approach."

This very welcome, thorough review article discusses and compares the various inter-related work of Bhaduri-Khare-Law, Berry-Keating, Aneva, Castro, *et.al.*, Pitkanen, Khuri, Joffily, Wu-Sprung, Okubo, Mussardo, Boos-Korepin, Crehan and others.

S. Davis, "Spin structures on Riemann surfaces and the perfect numbers" (preprint 12/1998)

"The equality between the number of odd spin structures on a Riemann surface of genus g, with $2^g - 1$ being a Mersenne prime, and the even perfect numbers, is an indication that the action of the modular group on the set of spin structures has special properties related to the sequence of perfect numbers. A method for determining whether Mersenne numbers are primes is developed by using a geometrical representation of these numbers. The connection between the non-existence of finite odd perfect numbers and the irrationality of the square root of twice the product of a sequence of repunits is investigated, and it is demonstrated, for an arbitrary number of prime factors, that the products of the corresponding repunits will not equal twice the square of a rational number."

Related work by S. Davis:

"A method for generating Mersenne primes and the extent of the sequence of even perfect numbers" (preprint, again involving spin structures and dynamical systems)

"A rationality condition for the existence of odd perfect numbers" (preprint 11/2000)

"A proof of the odd perfect number conjecture" (preprint 01/2004)

"A recursion relation for the number of Goldbach partitions of an even integer" (preprint)

P. Frampton and T. Kephart, "Mersenne primes, polygonal anomalies and string theory classification"

"It is pointed out that the Mersenne primes $M_p = 2^{p-1}$ and associated perfect numbers $M_p = 2^{p-1}M_p$ play a significant role in string theory; this observation may suggest a classification of consistent string theories."

P.H. Frampton and Y. Okada, "*p*-Adic string *N*-point function", *Phys. Rev. Lett. B* **60** (1988) 484-486

N. Efremov and N.V. Mitskievich, "A T_0-discrete universe model with five low-energy fundamental interactions and the coupling constants hierarchy" (preprint, 09/03)

[abstract:] "A quantum model of universe is constructed in which values of dimensionless coupling constants of the fundamental interactions (including the cosmological constant) are determined via certain topological invariants of manifolds forming finite ensembles of 3D Seifert fibrations. The characteristic values of the coupling constants are explicitly calculated as the set of rational numbers (up to the factor 2π) on the basis of a hypothesis that these values are proportional to the mean relative fluctuations of discrete volumes of manifolds in these ensembles. The discrete volumes are calculated using the standard Alexandroff procedure of constructing T_0-discrete spaces realized as nerves corresponding to characteristic canonical triangulations which are compatible with the Milnor representation of Seifert fibered homology spheres being the building material of all used 3D manifolds. **Moreover, the determination of all involved homology spheres is based on the first nine prime numbers (p_1 =2, ..., p_9=23).** The obtained hierarchy of coupling constants at the present evolution stage of universe well reproduces the actual hierarchy of the experimentally observed dimensionless low-energy coupling constants."

D.B. Grunberg, "Integrality of open instanton numbers" (preprint 05/03)

[abstract:] "We prove the integrality of the open instanton numbers in two examples of counting holomorphic disks on local Calabi-Yau threefolds: the resolved conifold and the degenerate *P* x *P*. Given the B-model superpotential, we extract by hand all Gromow-Witten invariants in the expansion of the A-model superpotential. The proof of their integrality relies on enticing congruences of binomial coefficients modulo powers of a prime. We also derive an expression for the factorial (p^k-1)! modulo powers of the prime *p*. We generalise two theorems of elementary number theory, by Wolstenholme and by Wilson."

P. D'Eath and G. Esposito, "The effect of boundaries in one-loop quantum cosmology"

P. D'Eath and G. Esposito, "Local Boundary Conditions for the Dirac Operator and One-Loop Quantum Cosmology"

P. D'Eath and G. Esposito, "Spectral boundary conditions in one-loop quantum cosmology"

"For fermionic fields on a compact Riemannian manifold with boundary one has a choice between local and non-local (spectral) boundary conditions. The one-loop prefactor in the Hartle-Hawking amplitude in quantum cosmology can then be studied using the generalized Riemann zeta-function formed from the squared eigenvalues of the four-dimensional fermionic operators."

A.P. de Almeida, F.T. Brandt and J. Frenkel, "Thermal matter and radiation in a gravitational field"

"We study the one-loop contributions of matter and radiation to the gravitational polarization tensor at finite temperatures. Using the analytically continued imaginary-time formalism, the contribution of matter is explicitly given to next-to-leading T^2 order. We obtain an exact form for the contribution of radiation fields, expressed in terms of generalized Riemann zeta functions."

V. V. Nesterenko and I. G. Pirozhenko, "Justification of the zeta function renormalization in rigid string model"

"A consistent procedure for regularization of divergences and for the subsequent renormalization of the string tension is proposed in the framework of the one-loop calculation of the interquark potential generated by the Polyakov-Kleinert string. In this way, a justification of the formal treatment of divergences by analytic continuation of the Riemann and Epstein-Hurwitz zeta functions is given. A spectral representation for the renormalized string energy at zero temperature is derived, which enables one to find the Casimir energy in this string model at nonzero temperature very easy."

A. Edery, "Multidimensional cut-off technique, odd-dimensional Epstein zeta functions and Casimir energy of massless scalar fields", submitted to *J. Physics A*

[abstract:] "Quantum fluctuations of massless scalar fields represented by quantum fluctuations of the quasiparticle vacuum in a zero-temperature dilute Bose-Einstein condensate may well provide the first experimental arena for measuring the Casimir force of a field other than the electromagnetic field. This would constitute a real Casimir force measurement - due to quantum fluctuations - in contrast to thermal fluctuation effects. We develop a multidimensional cut-off technique for calculating the Casimir energy of massless scalar fields in d-dimensional rectangular spaces with q large dimensions and $d-q$ dimensions of length L and generalize the technique to arbitrary lengths. We explicitly evaluate the multidimensional remainder and express it in a form that converges exponentially fast. Together with the compact analytical formulas we derive, the numerical results are exact and easy to obtain. Most importantly, we show that the division between analytical and remainder is not arbitrary but has a natural physical interpretation. The analytical part can be viewed as the sum of individual parallel plate energies and the remainder as an interaction energy. In a separate procedure, **via results from number theory, we express some odd-dimensional homogeneous Epstein zeta functions as products of one-dimensional sums plus a tiny remainder and calculate from them the Casimir energy via zeta function regularization.**"

V. Di Clemente, S. F. King and D.A.J. Rayner, "Supersymmetry and electroweak breaking with large and small extra dimensions", *Nucl. Phys. B* **617** (2001) 71-100

[abstract:] "We consider the problem of supersymmetry and electroweak breaking in a 5d theory compactified on an S^{1}/Z_{2} orbifold, where the extra dimension may be large or small. We consider the case of a supersymmetry breaking 4d brane located at one of the orbifold fixed points with the Standard Model gauge sector, third family and Higgs fields in the 5d bulk, and the first two families on a parallel 4d matter brane located at the other fixed point. We compute the Kaluza-Klein mass spectrum in this theory using a matrix technique which allows us to interpolate between large and small extra dimensions. We also consider the problem of electroweak symmetry breaking in this theory and localize the Yukawa couplings on the 4d matter brane spatially separated from the brane where supersymmetry is broken. We calculate the 1-loop effective potential using a zeta-function regularization technique, and find that the dominant top and stop contributions are separately finite. Using this result we find consistent electroweak symmetry breaking for a compactification scale {$ 1/R \approx 830$ GeV} and a lightest Higgs boson mass $m_{h} \approx 170$ GeV."

K. Roland, "Two- and three-loop amplitudes in covariant loop calculus", *Nuclear Physics B* **313** (1989) 432-446

[abstract:] "We study two- and three-loop vacuum amplitudes for the closed bosonic string. We compare two sets of expressions for the corresponding density on moduli space. One is based on the covariant reggeon loop calculus (where modular invariance is not manifest). The other is based on analytic geometry. We want to prove identity between the two sets of expressions. Quite apart from demonstrating modular invariance of the reggeon results, this would in itself be a remarkable mathematical feature. Identity is established to 'high' order in some moduli and exactly in others. The expansions reveal an essentially number-theoretic structure. Agreement is found only by exploiting the connection between the four Jacobi theta-functions and number theory."

J. L. Petersen, K. O. Roland and J. R. Sidenius, "Modular invariance and covariant loop calculus", *Physics Letters B* **205** (1988) 262-266

[abstract:] "The covariant loop calculus provides an efficient technique for computing explicit expressions for the density on moduli space corresponding to arbitrary (bosonic string) loop diagrams. Since modular invariance is not manifest, however, we carry out a detailed comparison with known explicit two- and three-loop results derived using analytic geometry (one loop is known to be okay). We establish identity to 'high' order in some moduli and exactly in others. Agreement is found as a result of various non-trivial cancellations, in part related to number theory. We feel our results provide very strong support for the correctness of the covariant loop calculus approach."

G. Heinrich, T. Huber, D. Maitre, "Master integrals for fermionic contributions to massless three-loop form factors" (preprint 12/07)

[abstract:] "In this letter we continue the calculation of master integrals for massless three-loop form factors by giving analytical results for those diagrams which are relevant for the fermionic contributions proportional to N_F^2, N_F*N, and N_F/N. Working in dimensional regularisation, we express one of the diagrams in a closed form which is exact to all orders in epsilon, containing Gamma-functions and hypergeometric functions of unit argument. In all other cases we derive multiple Mellin-Barnes representations from which the coefficients of the Laurent expansion in epsilon are extracted in an analytical form. To obtain the finite part of the three-loop quark and gluon form factors, all coefficients through transcendentality six in the Riemann zeta-function have to be included."

M.B. Green, J.G. Russo, P. Vanhove, "Low energy expansion of the four-particle genus-one amplitude in type II superstring theory" (preprint 01/2008)

[abstract:] "A diagrammatic expansion of coefficients in the low-momentum expansion of the genus-one four-particle amplitude in type II superstring theory is developed. This is applied to determine coefficients up to order s^6R^4 (where s is a Mandelstam invariant and R^4 the linearized super-curvature), and partial results are obtained beyond that order. This involves integrating powers of the scalar propagator on a toroidal world-sheet, as well as integrating over the modulus of the torus. At any given order in s the coefficients of these terms are given by rational numbers multiplying multiple zeta values (or Euler-Zagier sums) that, up to the order studied here, reduce to products of Riemann zeta values. We are careful to disentangle the analytic pieces from logarithmic threshold terms, which involves a discussion of the conditions imposed by unitarity. We further consider the compactification of the amplitude on a circle of radius r, which results in a plethora of terms that are power-behaved in r. These coefficients provide boundary `data' that must be matched by any non-perturbative expression for the low-energy expansion of the four-graviton amplitude.

The paper includes an appendix by Don Zagier. "

V. Ravindran, J. Smith and W.L. van Neerven, "Two-loop corrections to Higgs boson production" (Report-no: YITP-SB-04-46, 08/04)

[abstract:] "In this paper we present the two-loop vertex corrections to scalar and pseudo-scalar Higgs boson production for general colour factors for the gauge group SU(*N*) We derive a general formula for the vertex correction which holds for conserved and non conserved operators. For the conserved operator we take the electromagnetic vertex correction as an example whereas for the non conserved operators we take the two vertex corrections above. Our observations for the structure of the pole terms $1/\varepsilon^4$, $1/\varepsilon^3$ and $1/\varepsilon^2$ in two loop order are the same as made earlier in the literature for electromagnetism. For the single pole terms $1/\varepsilon$ we can predict the terms containing the Riemann zeta functions $\zeta(2)$ and $\zeta(3)$.

S.K. Ashok, F. Cachazo, E. Dell'Aquila, "Children's drawings from Seiberg-Witten curves", *Communications in Number Theory and Physics* **1** no. 2 (2007) 237-305

[abstract:] "We consider $N=2$ supersymmetric gauge theories perturbed by tree level superpotential terms near isolated singular points in the Coulomb moduli space. We identify the Seiberg-Witten curve at these points with polynomial equations used to construct what Grothendieck called "dessins d'enfants" or "children's drawings" on the Riemann sphere. From a mathematical point of view, the dessins are important because the absolute Galois group $Gal(\bar{Q}/Q)$ acts faithfully on them. We argue that the relation between the dessins and Seiberg-Witten theory is useful because gauge theory criteria used to distinguish branches of $N=1$ vacua can lead to mathematical invariants that help to distinguish dessins belonging to different Galois orbits. For instance, we show that the confinement index defined in hep-th/0301006 is a Galois invariant. We further make some conjectures on the relation between Grothendieck's programme of classifying dessins into Galois orbits and the physics problem of classifying phases of $N=1$ gauge theories. "

G. Moore, "Arithmetic and attractors" (preprint 07/03)

[abstract:] "We study relations between some topics in number theory and supersymmetric black holes. These relations are based on the "attractor mechanism" of *N*=2 supergravity. In IIB string compactification this mechanism singles out certain "attractor varieties". ' We show that these attractor varieties are constructed from products of elliptic curves with complex multiplication for *N*=4 and *N*=8 compactifications. The heterotic dual theories are related to rational conformal field theories. In the case of *N*=4 theories U-duality inequivalent backgrounds with the same horizon area are counted by the class number of a quadratic imaginary field. The attractor varieties are defined over fields closely related to class fields of the quadratic imaginary field. We discuss some extensions to more general Calabi-Yau compactifications and explore further connections to arithmetic including connections to Kronecker's Jugendtraum and the theory of modular heights. The paper also includes a short review of the attractor mechanism. A much shorter version of the paper summarizing the main points is the companion note entitled "Attractors and Arithmetic""

Yu. Manin and M. Marcolli, "Holography principle and arithmetic of algebraic curves", *Adv. Theor. Math. Phys.* **5** (2001), no. 3, 617–650.

[abstract:] "According to the holography principle (due to G. 't Hooft, L. Susskind, J. Maldacena, *et al.*), quantum gravity and string theory on certain manifolds with boundary can be studied in terms of a conformal field theory on the boundary. Only a few mathematically exact results corroborating this exciting program are known. In this paper we interpret from this perspective several constructions which arose initially in the arithmetic geometry of algebraic curves. We show that the relation between hyperbolic geometry and Arakelov geometry at arithmetic infinity involves exactly the same geometric data as the Euclidean AdS_3 holography of black holes. Moreover, in the case of Euclidean AdS_2 holography, we present some results on bulk/boundary correspondence where the boundary is a non-commutative space."

Yu. Manin, "Reflections on arithmetical physics", in *Conformal Invariance and String Theory* (Academic, 1989) 293–303.

M. Marcolli's survey article "Number Theory in Physics" contains some material on string theory.

C. Hattori, M. Matsunaga, T. Matsuoka, K. Nakanishi, "Galois group on elliptic curves and flavor symmetry" (preprint 10/07)

[abstract:] "Putting emphasis on the relation between rational conformal field theory (RCFT) and algebraic number theory, we consider a brane configuration in which the D-brane intersection is an elliptic curve corresponding to RCFT. A new approach to the generation structure of fermions is proposed in which the flavor symmetry including the R-parity has its origin in the Galois group on elliptic curves with complex multiplication (CM). We study the possible types of the Galois group derived from the torsion points of the elliptic curve with CM. A phenomenologically viable example of the Galois group is presented, in which the characteristic texture of fermion masses and mixings is reproduced and the mixed-anomaly conditions are satisfied."

C. Castro, "On the Riemann Hypothesis and tachyons in dual string scattering amplitudes", *International Journal of Geometric Methods in Modern Physics* **3** no. 2 (2006) 187-199

[abstract:] "It is the purpose of this work to pursue a novel physical interpretation of the nontrivial Riemann zeta zeros and prove why the location of these zeros $z_n = 1/2+i\lambda_n$ corresponds *physically* to tachyonic-resonances/tachyonic-condensates, originating from the scattering of two on-shell *tachyons* in bosonic string theory. Namely, we prove that if there were nontrivial zeta zeros (violating the Riemann hypothesis) outside the critical line Real $z = 1/2$ (but inside the critical strip), these putative zeros *do not* correspond to any *poles* of the bosonic open string scattering (Veneziano) amplitude $A(s,t,u)$. The *physical* relevance of tachyonic-resonances/tachyonic-condensates in bosonic string theory, establishes an important connection between string theory and the Riemann Hypothesis. In addition, one has also a *geometrical* interpretation of the zeta zeros in the critical line in terms of very special (degenerate) triangular configurations in the upper-part of the complex plane."

C. Castro (Perelman), "*p*-Adic stochastic dynamics, supersymmetry and the Riemann conjecture" [pdf file]

"Supersymmetry, *p*-adic stochastic dynamics, Brownian motion, Fokker-Planck equation, Langevin equation, prime number random distribution, random matrices, *p*-adic fractal strings, the adelic condition, etc...are all deeply interconnected in this paper."

C. Bachas and I. Brunner, "Fusion of conformal interfaces" (preprint 12/2007)

[abstract:] "We study the fusion of conformal interfaces in the c=1 conformal field theory. We uncover an elegant structure reminiscent of that of black holes in supersymmetric theories. The role of the BPS black holes is played by topological interfaces, which (a) minimize the entropy function, (b) fix through an attractor mechanism one or both of the bulk radii, and (c) are (marginally) stable under splitting. One significant difference is that the conserved charges are logarithms of natural numbers, rather than vectors in a charge lattice, as for BPS states. Besides potential applications to condensed-matter physics and number theory, these results point to the existence of large solution-generating algebras in string theory."

C. Lousto, "Towards the solution of the relativistic gravitational radiation reaction problem for binary black holes"

"Here we present the results of applying the generalized Riemann zeta-function regularization method to the gravitational radiation reaction problem. We analyze in detail the headon collision of two nonspinning black holes with extreme mass ratio. The resulting reaction force on the smaller hole is repulsive. We discuss the possible extensions of these method to generic orbits and spinning black holes. The determination of corrected trajectories allows to add second perturbative corrections with the consequent increase in the accuracy of computed waveforms."

S. Benvenuti, B. Feng, A. Hanany, Yang-Hui He, "BPS operators in gauge theories: Quivers, syzygies and plethystics" (preprint 08/2006)

[abstract:] "We develop a systematic and efficient method of counting single-trace and multi-trace BPS operators for world-volume gauge theories of N D-brane probes, for both N -> infinity and finite N. The techniques are applicable to generic singularities, orbifold, toric, non-toric, et cetera, even to geometries whose precise field theory duals are not yet known. Mathematically, fascinating and intricate inter-relations between gauge theory, algebraic geometry, combinatorics and number theory exhibit themselves in the form of plethystics and syzygies."

P.A. Perry and F.L. Williams, "Selberg zeta function and trace formula for the BTZ black hole" (preprint 02/03) [abstract:] "A Selberg zeta function is attached to the three-dimensional BTZ black hole, and a trace formula is developed for a general class of test functions. The trace formula differs from those of more standard use in physics in that the black hole has a fundamental domain of infinite hyperbolic volume. Various thermodynamic quantities associated with the black hole are conveniently expressed in terms of the zeta function."

A.L. Kholodenko, "Statistical mechanics of 2+1 gravity from Riemann zeta function and Alexander polynomial: Exact results"

"In the recent publication (*Journal of Geometry and Physics*, **33** (2000) 23-102) we demonstrated that dynamics of 2+1 gravity can be described in terms of train tracks. Train tracks were introduced by Thurston in connection with description of dynamics of surface automorphisms. In this work we provide an example of utilization of general formalism developed earlier. The complete exact solution of the model problem describing equilibrium dynamics of train tracks on the punctured torus is obtained. Being guided by similarities between the dynamics of 2d liquid crystals and 2+1 gravity the partition function for gravity is mapped into that for the Farey spin chain. The Farey spin chain partition function, fortunately, is known exactly and has been thoroughly investigated recently. Accordingly, the transition between the pseudo-Anosov and the periodic dynamic regime (in Thurston's terminology) in the case of gravity is being reinterpreted in terms of phase transitions in the Farey spin chain whose partition function is just a ratio of two Riemann zeta functions. The mapping into the spin chain is facilitated by recognition of a special role of the Alexander polynomial for knots/links in study of dynamics of self homeomorphisms of surfaces. At the end of paper, using some facts from the theory of arithmetic hyperbolic 3-manifolds (initiated by Bianchi in 1892), we develop systematic extension of the obtained results to noncompact Riemannian surfaces of higher genus. Some of the obtained results are also useful for 3+1 gravity. In particular, using the theorem of Margulis, we provide new reasons for the black hole existence in the Universe: black holes make our Universe arithmetic. That is the discrete Lie groups of motion are arithmetic."

J. Minahan, "Mode interactions of the tachyon condensate in *p*-adic string theory"

"We study the fluctuation modes for lump solutions of the tachyon effective potential in *p*-adic open string theory. We find a discrete spectrum with equally spaced mass squared levels. We also find that the interactions derived from this field theory are consistent with *p*-adic string amplitudes for excited string

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J.A. Nogueira, A. Maia, Jr., "Demonstration of how the zeta function method for effective potential removes the divergences"

[abstract:] "The calculation of the minimum of the effective potential using the zeta function method is extremely advantagous, because the zeta function is regular at $s = 0$ and we gain immediately a finite result for the effective potential without the necessity of subtraction of any pole or the addition of infinite counter-terms. The purpose of this paper is to explicitly point out how the cancellation of the divergences occurs and that the zeta function method implicitly uses the same procedure used by Bollini-Giambiagi and Salam-Strathdee in order to gain finite part of functions with a simple pole."

V.S. Vladimirov and Ya.I. Volovich, "On the nonlinear dynamical equation in the *p*-adic string theory" (preprint 06/03)

[abstract:] "In this work nonlinear pseudo-differential equations with the infinite number of derivatives are studied. These equations form a new class of equations which initially appeared in *p*-adic string theory. These equations are of much interest in mathematical physics and its applications in particular in string theory and cosmology. In the present work a systematical mathematical investigation of the properties of these equations is performed. The main theorem of uniqueness in some algebra of tempered distributions is proved. Boundary problems for bounded solutions are studied, the existence of a space-homogenous solution for odd *p* is proved. For even *p* it is proved that there is no continuous solutions and it is pointed to the possibility of existence of discontinuous solutions. Multidimensional equation is also considered and its soliton and *q*-brane solutions are discussed."

I. Ya. Aref'eva, I.V. Volovich, "Quantization of the Riemann zeta-function and cosmology" (preprint 12/2006)

[abstract:] "Quantization of the Riemann zeta-function is proposed. We treat the Riemann zeta-function as a symbol of a pseudodifferential operator and study the corresponding classical and quantum field theories. This approach is motivated by the theory of p-adic strings and by recent works on stringy cosmological models. We show that the Lagrangian for the zeta-function field is equivalent to the sum of the Klein-Gordon Lagrangians with masses defined by the zeros of the Riemann zeta-function. Quantization of the mathematics of Fermat-Wiles and the Langlands program is indicated. The Beilinson conjectures on the values of L-functions of motives are interpreted as dealing with the cosmological constant problem. Possible cosmological applications of the zeta-function field theory are discussed."

A. Sen, "Tachyon condensation and brane descent relations in *p*-adic string theory"

"It has been conjectured that an extremum of the tachyon potential of a bosonic D-brane represents the vacuum without any D-brane, and that various tachyonic lump solutions represent D-branes of lower dimension. We show that the tree level effective action of *p*-adic string theory, the expression for which is known exactly, provides an explicit realisation of these conjectures."

I.Ya. Aref'eva, M.G. Ivanov and I.V. Volovich, "Non-extremal intersecting *p*-branes in various dimensions", *Phys. Lett. B* **406** (1997) 44-48

[abstract:] "Non-extremal intersecting *p*-brane solutions of gravity coupled with several antisymmetric fields and dilatons in various space-time dimensions are constructed. The construction uses the same algebraic method of finding solutions as in the extremal case and a modified "no-force" conditions. We justify the "deformation" prescription. It is shown that the non-extremal intersecting *p*-brane solutions satisfy harmonic superposition rule and the intersections of non-extremal *p*-branes are specified by the same characteristic equations for the incidence matrices as for the extremal p-branes. We show that S-duality holds for non-extremal *p*-brane solutions. Generalized T-duality takes place under additional restrictions to the parameters of the theory which are the same as in the extremal case."

I.Ya.Arefeva, K.S.Viswanathan, A.I.Volovich and I.V.Volovich, "Composite *p*-branes in various dimensions", *Nucl. Phys. Proc. Suppl.* **56B** (1997) 52-60

[abstract:] "We review an algebraic method of finding the composite p-brane solutions for a generic Lagrangian, in arbitrary spacetime dimension, describing an interaction of a graviton, a dilaton and one or two antisymmetric tensors. We set the Fock-De Donder harmonic gauge for the metric and the "no-force" condition for the matter fields. Then equations for the antisymmetric field are reduced to the Laplace equation and the equation of motion for the dilaton and the Einstein equations for the metric are reduced to an algebraic equation. Solutions composed of *n* constituent *p*-branes with n independent harmonic functions are given. The form of the solutions demonstrates the harmonic functions superposition rule in diverse dimensions. Relations with known solutions in D = 10 and D = 11 dimensions are discussed."

I.Ya. Aref'eva, K.S. Viswanathan and I.V. Volovich, "*p*-Brane solutions in diverse dimensions", *Phys.Rev.* **D55** (1997) 4748-4755

[abstract:] "A generic Lagrangian, in arbitrary spacetime dimension, describing the interaction of a graviton, a dilaton and two antisymmetric tensors is considered. An isotropic *p*-brane solution consisting of three blocks and depending on four parameters in the Lagrangian and two arbitrary harmonic functions is obtained. For specific values of parameters in the Lagrangian the solution may be identified with previously known superstring solutions."

I.V. Volovich, "*p*-Adic string", *Classical Quantum Gravity* **4** (1987) 83-87

I.V. Volovich, "From *p*-adic strings to étale strings", *Proc. Steklov Inst. Math.* **203** (1995) no. 3, 37–42.

I.Aref'eva and A. Volovich, "Composite *p*-branes in diverse dimensions", *Class. Quant. Grav.* **14** (1997) 2991-3000 [abstract:] "We use a simple algebraic method to find a special class of composite *p*-brane solutions of higher dimensional gravity coupled with matter. These solutions are composed of *n* constituent *p*-branes corresponding n independent harmonic functions. A simple algebraic criteria of existence of such solutions is presented. Relations with *D* = 11, 10 known solutions are discussed."

A. Volovich, "Three-block p-branes in various dimensions", *Nucl. Phys.* **B492** (1997) 235-248

[abstract:] "It is shown that a Lagrangian, describing the interaction of the gravitation field with the dilaton and the antisymmetric tensor in arbitrary dimension spacetime, admits an isotropic *p*-brane solution consisting of three blocks. Relations with known *p*-brane solutions are discussed. In particular, in ten-dimensional spacetime the three-block *p*-brane solution is reduced to the known solution, which recently has been used in the D-brane derivation of the black hole entropy."

In Section 2.3 of Bernard Julia's seminal 1990 paper "Statistical theory of numbers", the author turns briefly from multiplicative to additive number theory, in particular to generating functionals associated with integer partition problems. He relates these to the Veneziano open string model, the tachyon mode, and the phenomenon of "bosonization" which is discussed elsewhere in the paper.

L. Brekke and P. Freund, "*p*-adic numbers in physics", *Physics Reports* **233**, (1993) 1-66.

This is a review article related to the achievements in application of *p*-adic numbers to string theory, quantum field theory and quantum mechanics during the period 1987-1992. The contribution of Freund and his collaborators is emphasised.

Here is an excerpt from pp61-62:

"String theory, the candidate "theory of everything" is expected to raise fundamental issues both at the level of physics and at the level of mathematics. The old issue of the nature of continuity in physics naturally leads to the consideration of *p*-adic strings. It is remarkable that these very simple alternate topologies have not appeared earlier in physics (ultrametrics have appeared [62]). Yet, even now it would not be reasonable to actually select a prime and claim this to be the phenomenologically preferred prime which "underlies" physics. As we have seen, such a preferred prime could lead to serious causality problems. But if none of the primes is to be preferred, then why select a priori the prime at infinity, and deal exclusively with real numbers? A more "even-handed" procedure would envision dealing with all primes at the same time. This naturally leads to adelic theories. We have seen that this point of view immediately yields the remarkable adelic product formulae. Could it be that the adelic string is the "real thing"? This question has been raised by Manin [41] in the following (somewhat paraphrased) form. Supposing that the true physics is adelic, then why can we always assume it to be archimedean, grounded in the real numbers? Maybe this is on account of some experimental limitations, e.g. low energy. Could it be that once these limitations get lifted and we reach very high (Planck) energies, the full adelic structure of the string will reveal itself? This is an interesting possibility.

Another possibility is that the true theory is archimedean, but that on account of the product formulae, one could alternatively conceive of the theory as an Euler product over all *p*-adic theories. As we saw, each such theory puts the strings' world sheet on a Bethe lattice. What the adelic formulae then tell us is that we should not opt for a particular Bethe lattice as the discretization of the world sheet, but rather study absolutely all of them. The cumulative understanding of all these discretizations is tantamount to understanding the ordinary archimedean string. Of course, each of these discretizations is far simpler than the ordinary string.

On the other hand, there is the *p*-adics-quantum group connection, which places the ordinary and all the *p*-adic strings at certain special points in a continuum of theories. It is an important problem to assess the theoretical consistency of all these "quantum" strings and the phenomenological possibilities offered by them."

M. Thomas, "Monster Sporadic Group encoding of the Schwarzschild metric" (preprint 09/04)

[abstract:] "The classic construction of the Monster Sporadic Group involves the 24 dimensional Leech lattice and a 2 dimensional orbifold which are related to the 26 dimensional bosonic string. It will be shown that there is an astrophysical model encoded with this largest of the finite sporadics with implications in quantum gravity and a possible physical mechanism for exploration of the 26 dimensional bosonic string. Precision calculations will be presented along with a heuristic approach to higher dimensional theory. Finally, the model in a highly excited state evaporates from a final fermion pair coupled to a Planck mass density bubble and due to a vanishingly small Planck's constant realizes the Einstein-Hilbert action going from larger degrees of freedom to the 4D low-energy state. This suggests that the unitarity of the quantum through the classical GRT is preserved."

Note that the elements of the Monster involve Ogg's supersingular primes. Here are some other instances of The Monster in string theory:

G. Chapline, "The Monster Sporadic Group and a theory underlying superstring models"

F. Lizzi and R.J. Szabo, "Duality symmetries and noncommutative geometry of string spacetime"

F. Lizzi and R.J. Szabo, "Noncommutative Geometry and Spacetime Gauge Symmetries of String Theory" , *Chaos, Solitons and Fractals* **10** (1999) 445-458

B. Craps, M.R. Gaberdiel and J.A. Harvey, "Monstrous branes", *Commun. Math. Phys.* **234** (2003) 229-251

M.B. Green and D. Kutasov, "Monstrous heterotic quantum mechanics", *Journal of High Energy Physics* **1** (01-12), 1-6

P. Henry-Labordere, B. Julia and L. Paulot, "Symmetries in M-theory: Monsters Inc." (talk given by PHL at Cargese 2002)

M.A.R. Osorio and M.A. Vazquez-Mozo, "Strings below the Planck scale", *Phys. Lett.* B **280** (1992) 21-25

J.A. Harvey and G. Moore, "Exact gravitational threshold correction in the FHSV model", *Phys. Rev.* D **57** (1998) 2329-2336

S. Chaudhuri and D.A. Lowe, "Monstrous string-string duality", *Nucl. Phys.* B **469** (1996) 21-36

F.D.T. Smith, "E6, strings, branes, and the standard model" (preprint 04/04)

L. Dolan and M. Langham, "Symmetric subgroups of gauged supergravities and AdS string theory vertex operators", *Mod. Phys. Lett.* A **14** (1999) 517-526

L. Dolan, "The beacon of Kac-Moody symmetry for physics", *Notices of the American Mathematical Society*, Dec. 1995, 1489

P.C. West, "Physical states and string symmetries", *Mod. Phys. Lett.* A **10** (1995) 761-772

M. Lapidus, *In Search of the Riemann Zeros* (AMS, 2008)

[from publisher's description:] "In this book, the author proposes a new approach to understand and possibly solve the Riemann Hypothesis. His reformulation builds upon earlier (joint) work on complex fractal dimensions and the vibrations of fractal strings, combined with string theory and noncommutative geometry. Accordingly, it relies on the new notion of a fractal membrane or quantized fractal string, along with the modular flow on the associated modui space of fractal membranes. Conjecturally, under the action of the modular flow, the spacetime geometries become increasingly symmetric and crystal-like, hence, arithmetic. Correspondingly, the zeros of the associated zeta functions eventually condense onto the critical line, towards which they are attracted, thereby explaining why the Riemann Hypothesis must be true.

Written with a diverse audience in mind, this unique book is suitable for graduate students, experts and nonexperts alike, with an interest in number theory, analysis, dynamical systems, arithmetic, fractal or noncommutative geometry, and mathematical and mathematical or theoretical physics."

Conference: "Modular Forms and String Duality", Banff International Research Station, June 3-8, 2006

"Physical duality symmetries relate special limits of the various consistent string theories (Types I, II, Heterotic string and their cousins, including F-theory) one to another. By comparing the mathematical descriptions of these theories, one reveals often quite deep and unexpected mathematical conjectures. The best known string duality to mathematicians, Type IIA/IIB duality also called *mirror symmetry*, has inspired many new developments in algebraic and arithmetic geometry, number theory, toric geometry, Riemann surface theory, and infinite dimensional Lie algebras. Other string dualities such as Heterotic/Type II duality and F-Theory/Heterotic string duality have also, more recently, led to series of mathematical conjectures, many involving elliptic curves, K3 surfaces, and modular forms. Modular forms and quasi-modular forms play a central role in mirror symmetry, in particular, as generating functions counting the number of curves on Calabi-Yau manifolds and describing Gromov-Witten invariants. This has led to a realization that time is ripe to assess the role of number theory, in particular, that of modular forms, in mirror symmetry and string dualities in general.

One of the principal goals of this workshop is to look at modular and quasi-modular forms, congruence zeta-functions, Galois representations, and *L*-series for dual families of Calabi-Yau varieties with the aim of interpreting duality symmetries in terms of arithmetic invariants associated to the varieties in question. Over the last decades, a great deal of work has been done on these problems. In particular it appears that we need to modify the classical theories of Galois representations (in particular, the question of modularity) and modular forms, among others, for families of Calabi-Yau varieties in order to accommodate "quantum corrections"."

M. Nardelli, "Proposta di dimostrazione della variante Riemann di Lagarias (RH1), equivalente all'ipotesi di Riemann, con RH1=RH" (preprint in Italian, 12/2007)

[translation of abstract provided by author:] "In this paper, we suggest a proof of the Riemann Hypothesis by the 'Lagarias variant' or 'Lagarias Equivalence': RH1 = RH. Hence, we prove that for abundant and colossally abundant numbers, $L(n)$ increases progressively, as n increases, although with small oscillations, but which never lead to $L(n)$ taking negative or zero values.

Furthermore, we obtain also some interesting mathematical connections between various equations concerning the Riemann zeta function and some solutions of equations regarding various models of string cosmology."

M. Nardelli, "On the link between the structure of A-branes observed in homological mirror symmetry and the classical theory of automorphic forms. Mathematical connections with the modular elliptic curves, p-adic and adelic numbers and p-adic and adelic strings" (preprint 03/2008)

[abstract:] "This paper is a review of some interesting results that has been obtained in the study of the categories of A-branes on the dual Hitchin fibers and some interesting phenomena associated with the endoscopy in the geometric Langlands correspondence of various authoritative theoretical physicists and mathematicians."

M. Nardelli, "On some mathematical connections concerning the three-dimensional pure quantum gravity with negative cosmological constant, the Selberg zeta-function, the ten-dimensional anomaly cancellations, the vanishing of cosmological constant, and some sectors of string theory and number theory" (preprint 06/2008)

[abstract:] "This paper is a review of some interesting results that has been obtained in the study of the quantum gravity partition functions in three-dimensions, in the Selberg zeta function, in the vanishing of cosmological constant and in the ten-dimensional anomaly cancellations. In the Section 1, we have described some equations concerning the pure three-dimensional quantum gravity with a negative cosmological constant and the pure three-dimensional supergravity partition functions. In the Section 2, we have described some equations concerning the Selberg super-trace formula for Super-Riemann surfaces, some analytic properties of Selberg super zeta-functions and multiloop contributions for the fermionic strings. In the Section 3, we have described some equations concerning the ten-dimensional anomaly cancellations and the vanishing of cosmological constant. In the Section 4, we have described some equations concerning p-adic strings, p-adic and adelic zeta functions and zeta strings. In conclusion, in the Section 5, we have described the possible and very interesting mathematical connections obtained between some equations regarding the various sections and some sectors of number t heory (Riemann zeta functions, Ramanujan modular equations, etc…) and some interesting mathematical applications concerning the Selberg super-zeta functions and some equations regarding the Section 1."

Michele Nardelli has also provided four other preprints involving work relating aspects of number theory to string theory, quantum cosmology, gauge theory, noncommutative geometry, etc.: [1]   [2]   [3]   [4]

Here is an excerpt from a posting by on the sci.physics newsgroup (02/98) by Dan Piponi:

"In (bosonic) string theory via the operator formalism you find an infinite looking zero point energy just like in QED except that you get a sum that looks like:

$$1+2+3+4+...$$

Now the naive thing to do is the same: subtract off this zero point energy. However later on you get into complications. In fact (if I remember correctly) you must replace this infinity with -1/12 (of all things!) to keep things consistent.

Now it turns out there is a nice mathematical kludge that allows you to see 1+2+3+4+... as equalling -1/12. What you do is rewrite it as

$$1+2^{-n}+3^{-n}+...$$

This is the Riemann Zeta function. This converges for large n but can be analytically continued to $n = -1$, even though the series doesn't converge there. Zeta(-1) is -1/12. So in some bizarre sense 1+2+3+4+... really is -1/12.

But even more amazingly is that you can get the -1/12 by a completely different route - using the path integral formalism rather than the operator formalism. This -1/12 is tied up in a deep way with the geometry of string theory so it's a lot more than simply a trick to keep the numbers finite.

However I don't know if the equivalent operation in QED is tied up with the same kind of interesting geometry."

supersymmetry and number theory

A. Kapustin and E. Witten, "Electric-Magnetic Duality And The Geometric Langlands Program", *Communications in Number Theory and Physics* **1** no. 1 (2007) 1-236

[abstract:] "The geometric Langlands program can be described in a natural way by compactifying on a Riemann surface C a twisted version of $N=4$ super Yang-Mills theory in four dimensions. The key ingredients are electric-magnetic duality of gauge theory, mirror symmetry of sigma-models, branes, Wilson and 't Hooft operators, and topological field theory. Seemingly esoteric notions of the geometric Langlands program, such as Hecke eigensheaves and D-modules, arise naturally from the physics."

S.K. Ashok, F. Cachazo, E. Dell'Aquila, "Children's drawings from Seiberg-Witten curves", *Communications in Number Theory and Physics* **1** no. 2 (2007) 237-305

[abstract:] "We consider $N=2$ supersymmetric gauge theories perturbed by tree level superpotential terms near isolated singular points in the Coulomb moduli space. We identify the Seiberg-Witten curve at these points with polynomial equations used to construct what Grothendieck called "dessins d'enfants" or "children's drawings" on the Riemann sphere. From a mathematical point of view, the dessins are important because the absolute Galois group $Gal(\bar{Q}/Q)$ acts faithfully on them. We argue that the relation between the dessins and Seiberg-Witten theory is useful because gauge theory criteria used to distinguish branches of $N=1$ vacua can lead to mathematical invariants that help to distinguish dessins belonging to different Galois orbits. For instance, we show that the confinement index defined in hep-th/0301006 is a Galois invariant. We further make some conjectures on the relation between Grothendieck's programme of classifying dessins into Galois orbits and the physics problem of classifying phases of $N=1$ gauge theories. "

G. Chalmers, "Comment on the Riemann hypothesis" (preprint 03/05)

[abstract:] "The Riemann hypothesis is identified with zeros of ${\cal N}=4$ supersymmetric gauge theory four-point amplitude. The zeros of the $\zeta(s)$ function are identified with the complex dimension of the spacetime, or the dimension of the toroidal compactification. A sequence of dimensions are identified in order to map the zeros of the amplitude to the Riemann hypothesis."

D. Spector, "Duality, partial supersymmetry, and arithmetic number theory", *Journal of Mathematical Physics* **39**(4) (1998) 1919-1927

"We find examples of duality among quantum theories that are related to arithmetic functions by identifying distinct Hamiltonians that have identical partition functions at suitably related coupling constants or temperatures. We are led to this after first developing the notion of partial supersymmetry, in which some, but not all, of the operators of a theory have superpartners, and using it to construct fermionic and parafermionic thermal partition functions, and to derive some number theoretic identities. In the process, we also find a bosonic analog of the Witten index, and use this, too, to obtain some number theoretic results related to the Riemann zeta function."

D. Spector, "Supersymmetry and the Mobius inversion function", *Communications in Mathematical Physics* **127** (1990) 239.

"We show that the Mobius inversion function of number theory can be interpreted as the operator $(-1)^F$ in quantum field theory...We will see in this paper that the function...has a very natural interpretation. In the proper context, it is equivalent to $(-1)^F$, the operator that distinguishes fermionic from bosonic states and operators, with the fact that *mu*(*n*) = 0 when *n* is not squarefree being equivalent to the Pauli exclusion principle...One of the results we obtain is equivalent to the prime number theorem, one of the central achievements of number theory, in which the asymptotic density of prime numbers is computed."

C. Castro, "On two strategies towards the Riemann Hypothesis: Fractal Supersymmetric QM and a trace formula" (preprint 06/06)

[abstract:] "The Riemann Hypothesis (RH) states that the nontrivial zeros of the Riemann zeta-function are of the form $s_n =1/2+i lambda_n$. An improvement of our previous construction to prove the RH is presented by implementing the Hilbert-Pólya proposal and furnishing the Fractal Supersymmetric Quantum Mechanical (SUSY-QM) model whose spectrum reproduces the imaginary parts of the zeta zeros. We model the fractal fluctuations of the smooth Wu-Sprung potential (that capture the average level density of zeros) by recurring to a weighted superposition of Weierstrass functions $W(x,p,D)$ and where the summation has to be performed over all primes p in order to recapture the connection between the distribution of zeta zeros and prime numbers. We proceed next with the construction of a smooth version of the fractal QM wave equation by writing an ordinary Schrödinger equation whose fluctuating potential (relative to the smooth Wu-Sprung potential) has the same functional form as the fluctuating part of the level density of zeros. The second approach to prove the RH relies on the existence of a continuous family of scaling-like operators involving the Gauss-Jacobi theta series. An explicit trace formula related to a superposition of eigenfunctions of these scaling-like operators is defined. If the trace relation is satisfied this could be another test of the Riemann Hypothesis."

C. Castro (Perelman), "*p*-Adic stochastic dynamics, supersymmetry and the Riemann conjecture"

"Supersymmetry,*p*-adic stochastic dynamics, Brownian motion, Fokker-Planck equation, Langevin equation, prime number random distribution, random matrices, p-adic fractal strings, the adelic condition, etc...are all deeply interconnected in this paper."

C. Castro, A. Granik, and J. Mahecha, "On SUSY-QM, fractal strings and steps towards a proof of the Riemann hypothesis"

"The steps towards a proof of Riemann's conjecture using spectral analysis are rigorously provided. We prove that the only zeroes of the Rieamnn zeta-function are of the form $s = 1/2 + i\ lambda_n$. A supersymmetric quantum mechanical model is proposed as an alternative way to prove the Riemann conjecture, inspired in the Hilbert-Pólya proposal; it uses an inverse scattering approach associated with a system of p-adic harmonic oscillators. An interpretation of the Riemann's fundamental relation $Z(s) = Z(1 - s)$ as a duality relation, from one fractal string L to another dual fractal string L' is proposed."

C. Castro and J. Mahecha, "A fractal SUSY-QM model and the Riemann hypothesis" (preprint 06/03)

[abstract:] "The Riemann hypothesis (RH) states that the nontrivial zeros of the Riemann zeta-function are of the form $s = 1/2 + i\lambda_n$. Hilbert-Polya argued that if a Hermitian operator exists whose eigenvalues are the imaginary parts of the zeta zeros, λ_n, then the RH is true. In this paper a fractal supersymmetric quantum mechanical (SUSY-QM) model is proposed to prove the RH. It is based on a quantum inverse scattering method related to a fractal potential given by a Weierstrass function (continuous but nowhere differentiable) that is present in the fractal analog of the CBC (Comtet, Bandrauk, Campbell) formula in SUSY QM. It requires using suitable fractal derivatives and integrals of irrational order whose parameter β is one-half the fractal dimension of the Weierstrass function. An ordinary SUSY-QM oscillator is constructed whose eigenvalues are of the form $\lambda_n = n\pi$, and which coincide with the imaginary parts of the zeros of the funciton $\sin(iz)$. This sine function obeys a trivial analog of the RH. A review of our earlier proof of the RH based on a SUSY QM model whose potential is related ot the Gauss-Jacobi theta series is also included. The spectrum is given by $s(1 - s)$ which is real in the critical line (location of the nontrivial zeros) and in the real axis (location of the trivial zeros)."

C. Castro and J. Mahecha, "Fractal supersymmetric quantum mechanics, geometric probability and the Riemann Hypothesis", *International Journal of Geometric Methods in Modern Physics* **1** no. 6 (2004) 751-793

[abstract:] "The Riemann Hypothesis (RH) states that the nontrivial zeros of the Riemann zeta-function are of the form $s = 1/2 + i\lambda_{n}$. Earlier work on the RH based on Supersymmetric QM, whose potential was related to the Gauss-Jacobi theta series, allows to provide the proper framework to construct the well defined algorithm to compute the probability to find a zero (an infinity of zeros) in the critical line. Geometric Probability Theory furnishes the answer to the very difficult question whether the *probability* that the

RH is true is indeed equal to *unity* or not. To test the validity of this Geometric Probabilistic framework to compute the probability if the RH is true, we apply it directly to the hyperbolic sine function *sinh*(*s*) case which obeys a trivial analog of the RH. Its zeros are equally spaced in the imaginary axis $s_n = 0 + in\pi$. The Geometric Probability to find *a* zero (and an infinity of zeros) in the imaginary axis is exactly *unity*. We proceed with a fractal supersymmetric quantum mechanical (SUSY-QM) model to implement the Hilbert-Pólya proposal to prove the RH by postulating a Hermitian operator that reproduces all the λ_n's for its spectrum. Quantum inverse scattering methods related to a *fractal* potential given by a Weierstrass function (continuous but nowhere differentiable) are applied to the analog of the fractal analog of the CBC (Comtet-Bandrauk-Campbell) formula in SUSY QM. It requires using suitable fractal derivatives and integrals of irrational order whose parameter β is one-half the fractal dimension (*D* = 1.5) of the Weierstrass function. An ordinary SUSY-QM oscillator is also constructed whose eigenvalues are of the form $\lambda_n = n\pi$ and which coincide which the imaginary parts of the zeros of the function *sinh*(*s*). Finally, we discuss the relationship to the theory of 1/*f* noise."

P.B. Slater, "A numerical examination of the Castro-Mahecha supersymmetric model of the Riemann zeros" (preprint 11/05)

[abstract:] "The unknown parameters of the recently-proposed (*Int J. Geom. Meth. Mod. Phys.* **1**, 751 [2004]) Castro-Mahecha model of the imaginary parts (lambda_{j}) of the nontrivial Riemann zeros are the phases (alpha_{k}) and the frequency parameter (gamma) of the Weierstrass function of fractal dimension D=3/2 and the turning points (x_{j}) of the supersymmetric potential-squared Phi^2(x) - which incorporates the smooth Wu-Sprung potential (*Phys. Rev. E* **48**, 2595 [1993]), giving the average level density of the Riemann zeros. We conduct numerical investigations to estimate/determine these parameters - as well as a parameter (sigma) we introduce to scale the fractal contribution. Our primary analyses involve two sets of coupled equations: one set being of the form Phi^{2}(x_{j}) = lambda_{j}, and the other set corresponding to the fractal extension - according to an ansatz of Castro and Mahecha - of the Comtet-Bandrauk-Campbell (CBC) quasi-classical quantization conditions for good supersymmetry. Our analyses suggest the possibility strongly that gamma converges to its theoretical lower bound of 1, and the possibility that all the phases (alpha_{k}) should be set to zero."

Y. Fyodorov, "Negative moments of characteristic polynomials of random matrices: Ingham-Siegel integral as an alternative to Hubbard-Stratonovich transformation", *Nuclear Physics B* **621** (2002) 643-674.

"We reconsider the problem of calculating arbitrary negative integer moments of the (regularized) characteristic polynomial for *N* x *N* random matrices taken from the Gaussian Unitary Ensemble (GUE). A very compact and convenient integral representation is found via the use of a matrix integral close to that considered by Ingham and Siegel. We find the asymptotic expression for the discussed moments in the limit of large *N*. The latter is of interest because of a conjectured relation to properties of the Riemann zeta-function zeroes. Our method reveals a striking similarity between the structure of the negative and positive integer moments which is usually obscured by the use of the Hubbard-Stratonovich transformation. This sheds a new light on "bosonic" versus "fermionic" replica trick and has some implications for the supersymmetry method. We briefly discuss the case of the chiral GUE model from that perspective."

M. Bordag, A. S. Goldhaber, P. van Nieuwenhuizen and D. Vassilevich, "Heat kernels and zeta-function regularization for the mass of the SUSY kink"

[abstract:] "We apply zeta-function regularization to the kink and susy kink and compute its quantum mass. We fix ambiguities by the renormalization condition that the quantum mass vanishes as one lets the mass gap tend to infinity while keeping scattering data fixed. As an alternative we write the regulated sum over zero point energies in t erms of the heat kernel and apply standard heat kernel subtractions. Finally we discuss to what extent these procedures are equivalent to the usual renormalization conditions that tadpoles vanish."

A. Pérez, M.R. De Traubenberg and P. Simon, "2D-fractional supersymmetry: from rational to irrational conformal field theory"

[abstract:] "Supersymmetry can be consistently generalized in one and two dimensional spaces, fractional supersymmetry being one of the possible extension. Fractional supersymmetry of arbitrary order F is explicitly constructed using an adapted superspace formalism. This symmetry connects the fractional spin states (0,{1 \over F}, \cdots,{ F-1 \over F}). Besides the stress momentum tensor, we obtain a conserved current of spin (1 + { 1 \over F}). The central charges are generally irrational numbers except for the particular cases F=2,3,4,6. A natural classification emerges according to the decomposition of F into its product of prime numbers leading to sub-systems with smaller symmetries. The limit F goes to the infinity is also considered."

D.B. Grunberg, "Integrality of open instanton numbers" (preprint, 05/03)

[abstract:] "We prove the integrality of the open instanton numbers in two examples of counting holomorphic disks on local Calabi-Yau threefolds: the resolved conifold and the degenerate *P* x *P*. Given the B-model superpotential, we extract by hand all Gromow-Witten invariants in the expansion of the A-model superpotential. The proof of their integrality relies on enticing congruences of binomial coefficients modulo powers of a prime. We also derive an expression for the factorial (p^k-1)! modulo powers of the prime *p*. We generalise two theorems of elementary number theory, by Wolstenholme and by Wilson."

V. Di Clemente, S. F. King and D.A.J. Rayner, "Supersymmetry and electroweak breaking with large and small extra dimensions", *Nucl. Phys. B* **617** (2001) 71-100

[abstract:] "We consider the problem of supersymmetry and electroweak breaking in a 5d theory compactified on an S^{1}/Z_{2} orbifold, where the extra dimension may be large or small. We consider the case of a supersymmetry breaking 4d brane located at one of the orbifold fixed points with the Standard Model gauge sector, third family and Higgs fields in the 5d bulk, and the first two families on a parallel 4d matter brane located at the other fixed point. We compute the Kaluza-Klein mass spectrum in this theory using a matrix technique which allows us to interpolate between large and small extra dimensions. We also consider the problem of electroweak symmetry breaking in this theory and localize the Yukawa couplings on the 4d matter brane spatially separated from the brane where supersymmetry is broken. We calculate the 1-loop effective potential using a zeta-function regularization technique, and find that the dominant top and stop contributions are separately finite. Using this result we find consistent electroweak symmetry breaking for a compactification scale {$ 1/R \approx 830$ GeV} and a lightest Higgs boson mass $m_{h} \approx 170$ GeV."

G. Moore, "Arithmetic and attractors" (preprint 07/03)

[abstract:] "We study relations between some topics in number theory and supersymmetric black holes. These relations are based on the "attractor mechanism" of *N*=2 supergravity. In IIB string compactification this mechanism singles out certain "attractor varieties". ' We show that these attractor varieties are constructed from products of elliptic curves with complex multiplication for *N*=4 and *N*=8 compactifications. The heterotic dual theories are related to rational conformal field theories. In the case of *N*=4 theories U-duality inequivalent backgrounds with the same horizon area are counted by the class number of a quadratic imaginary field. The attractor varieties are defined over fields closely related to class fields of the quadratic imaginary field. We discuss some extensions to more general Calabi-Yau compactifications and explore further connections to arithmetic including connections to Kronecker's Jugendtraum and the theory of modular heights. The paper also includes a short review of the attractor mechanism. A much shorter version of the paper summarizing the main points is the companion note entitled "Attractors and Arithmetic""

M. Nardelli, "On the possible mathematical connections between the Hartle-Hawking no boundary proposal concerning the Randall-Sundrum cosmological scenario, Hartle-Hawking wave-function in the mini-superspace of physical superstring theory, *p*-adic Hartle-Hawkind wave function and some sectors of number theory" (preprint, 2007)

M. Nardelli, "On the possible mathematical connections concerning noncommutative minisuperspace cosmology, noncommutative quantum cosmology in low-energy string action, noncommutative Kantowsky-Sachs quantum model, spectral action principle associated with a noncommutative space and some aspects concerning the loop quantum gravity" (preprint, 2007)

M. Nardelli, "On some mathematical connections concerning the three-dimensional pure quantum gravity with negative cosmological constant, the Selberg zeta-function, the ten-dimensional anomaly cancellations, the vanishing of cosmological constant, and some sectors of string theory and number theory" (preprint 06/2008)

[abstract:] "This paper is a review of some interesting results that has been obtained in the study of the quantum gravity partition functions in three-dimensions, in the Selberg zeta function, in the vanishing of cosmological constant and in the ten-dimensional anomaly cancellations. In the Section 1, we have described some equations concerning the pure three-dimensional quantum gravity with a negative cosmological constant and the pure three-dimensional supergravity partition functions. In the Section 2, we have described some equations concerning the Selberg super-trace formula for Super-Riemann surfaces, some analytic properties of Selberg super zeta-functions and multiloop contributions for the fermionic strings. In the Section 3, we have described some equations concerning the ten-dimensional anomaly cancellations and the vanishing of cosmological constant. In the Section 4, we have described some equations concerning p-adic strings, p-adic and adelic zeta functions and zeta strings. In conclusion, in the Section 5, we have described the possible and very interesting mathematical connections obtained between some equations regarding the various sections and some sectors of number t heory (Riemann zeta functions, Ramanujan modular equations, etc…) and some interesting mathematical applications concerning the Selberg super-zeta functions and some equations regarding the Section 1."

noncommutative geometry and number theory

Paul Smith's excellent "Noncommutative Geometry and Algebra" page

M. Marcolli, "Lectures on arithmetic noncommutative geometry"

[abstract:] "This is the text of a series of five lectures given by the author at the "Second Annual Spring Institute on Noncommutative Geometry and Operator Algebras" held at Vanderbilt University in May 2004. It is meant as an overview of recent results illustrating the interplay between noncommutative geometry and arithmetic geometry/number theory."

C. Consani and M. Marcolli (eds.), *Noncommutative Geometry and Number Theory: Where Arithmetic meets Geometry and Physics* (Vieweg Verlag, 2006)

[AMS website description:] "In recent years, number theory and arithmetic geometry have been enriched by new techniques from noncommutative geometry, operator algebras, dynamical systems, and K-Theory. This volume collects and presents up-to-date research topics in arithmetic and noncommutative geometry and ideas from physics that point to possible new connections between the fields of number theory, algebraic geometry and noncommutative geometry. The articles collected in this volume present new noncommutative geometry perspectives on classical topics of number theory and arithmetic such as modular forms, class field theory, the theory of reductive p-adic groups, Shimura varieties, the local L-factors of arithmetic varieties. They also show

how arithmetic appears naturally in noncommutative geometry and in physics, in the residues of Feynman graphs, in the properties of noncommutative tori, and in the quantum Hall effect."

Alain Connes-related material

A. Connes, "Noncommutative geometry and the Riemann zeta function" from *Mathematics: Frontiers and Perspectives 2000*, V. Arnold *et.al.*, eds. (2000) 35-55.

A. Connes, "Trace formula in noncommutative geometry and the zeros of the Riemann zeta function", *Selecta Math.* (N.S.) **5** (1999) 29-106.

A. Connes, "Noncommutative geometry year 2000", *GAFA* special volume 2000 (2001)

[Abstract:] "We describe basic concepts of noncommutative geometry and a general construction extending the familiar duality between ordinary spaces and commutative algebras to a duality between Quotient spaces and Noncommutative algebras. Basic tools of the theory, K-theory, Cyclic cohomology, Morita equivalence, Operator theoretic index theorems, Hopf algebra symmetry are reviewed. They cover the global aspects of noncommutative spaces, such as the transformation $\theta \to 1/\theta$ for the NC torus $\Tb_{\theta}^2$, unseen in perturbative expansions in θ such as star or Moyal products. We discuss the foundational problem of "what is a manifold in NCG" and explain the role of Poincare duality in K-homology which is the basic reason for the spectral point of view. When specializing to 4-geometries this leads to the universal "Instanton algebra". We describe our work with G. Landi which gives NC-spheres S_{θ}^4 from representations of the Instanton algebra. We show that any compact Riemannian spin manifold whose isometry group has rank $r \geq 2$ admits isospectral deformations to noncommutative geometries. We give a survey of our work with H. Moscovici on the transverse geometry of foliations which yields a diffeomorphism invariant geometry on the bundle of metrics on a manifold and a natural extension of cyclic cohomology to Hopf algebras. Then, our work with D. Kreimer on renormalization and the Riemann-Hilbert problem. Finally we describe the spectral realization of zeros of zeta and L-functions from the noncommutative space of Adele classes on a global field and its relation with the Arthur-Selberg trace formula in the Langlands program. We end with a tentalizing connection between the renormalization group and the missing Galois theory at Archimedian places."

A. Connes, "Formule de trace en geometrie non commutative et hypothese de Riemann", C.R.Sci. Paris, t.323, Serie 1 (Analyse) (1996) 1231-1235.;

[Abstract:] "We reduce the Riemann hypothesis for *L*-functions on a global field *k* to the validity (not rigorously justified) of a trace formula for the action of the idele class group on the noncommutative space quotient of the adeles of *k* by the multiplicative group of *k*."

P. Cohen "Dedekind zeta functions and quantum statistical mechanics"

[excerpt:] "[We] construct a quantum dynamical system with partition function the Riemann zeta function, or the Dedekind zeta function in the general number field case. In order for the quantum dynamical system to reflect the arithmetic of the primes it must capture also some sort of interaction between them. This last feature translates in the statistical mechanical language into the phenomenon of spontaneous symmetry breaking at critical temperature with respect to a natural symmetry group. In the region of high temperature, there is a unique equilibrium state as the system is in disorder and symmetric with respect to a natural symmetry group. In the region of low temperature, a phase transition occurs and the symmetry is broken. This symmetry group acts transitively on a family of possible extremal equilibrium states. The construction of a quantum dynamical system with partition function the Riemann zeta function *zeta*(*beta*) and spontaneous symmetry breaking or phase transition at its pole *beta* = 1 with respect to a natural symmetry group was achieved by Bost and Connes in [BC].

A different construction of the basic algebra using crossed products was proposed by Laca and Raeburn and extended to the number field case by them with Arledge in [ALR].

An extension of the work of Bost and Connes to general global fields was done by Harari and Leichtnam in [HL]. The generalisation proposed by Harari and Leichtnam in [HL] fails to capture the Dedekind zeta function as partition function in the case of a number field with class number greater than 1. Their partition function in that case is the Dedekind zeta function with a finite number of non-canonically chosen Euler factors removed. This prompted the author's paper [Coh1] where the full Dedekind zeta function is recovered as partition function. This is achieved by recasting the original construction of Bost and Connes more completely in terms of adeles and ideles."

[BC] J.-B. Bost and A. Connes, "Hecke Algebras, Type III factors and phase transitions with spontaneous symmetry breaking in number theory", Selecta Math. (New Series), **1** (1995) 411-457.

"In this paper, we construct a natural *C**-dynamical system whose partition function is the Riemann zeta function. Our construction is general and associates to an inclusion of rings (under a suitable finiteness assumption) an inclusion of discrete groups (the associated *ax + b* groups) and the corresponding Hecke algebras of bi-invariant functions. The latter algebra is endowed with a canonical one parameter group of automorphisms measuring the lack of normality of the subgroup. The inclusion of rings **Z** provides the desired *C**-dynamical system, which admits the zeta function as partition function and the Galois group Gal($\mathbf{Q}^{cycl}$/ **Q**) of the cyclotomic extension $\mathbf{Q}^{cycl}$ of **Q** as symmetry group. Moreover, it exhibits a phase transition with spontaneous symmetry breaking at inverse temperature *beta* = 1. The original motivation for these results comes from the work of B. Julia [J] (cf. also [Spe])."

[excerpt from p.413:] "We shall now describe (the precise motivation will be explained below) a C* dynamical system intimately related to the distribution of prime numbers and exhibiting the above behaviour of spontaneous symmetry breaking."

[ALR] J. Arledge, M. Laca, I. Raeburn, "Semigroup crossed products and Hecke algebras arising from number fields", *Doc. Mathematica* **2** (1997) 115-138.

[HL] D. Harari and E. Leichtnam "Extension du phenomene de brisure spontanee de symetrie de Bost-Connes au cas des corps global quelconques"

[Coh1] P.B. Cohen, "A C*-dynamical system with Dedekind zeta partition function and spontaneous symmetry breaking", soumis aux Actes des Journees Arithmetiques de Limoges, 1997. Preprint de l'IRMA de l'UST de Lille.

A. Connes and M. Marcolli, "From Physics to Number Theory via Noncommutative Geometry. Part I: Quantum Statistical Mechanics of **Q**-lattices" (preprint 04/04)

[abstract:] "This is the first installment of a paper in three parts, where we use noncommutative geometry to study the space of commensurability classes of **Q**-lattices and we show that the arithmetic properties of KMS states in the corresponding quantum statistical mechanical system, the theory of modular Hecke algebras, and the spectral realization of zeros of L-functions are part of a unique general picture. In this first chapter we give a complete description of the multiple phase transitions and arithmetic spontaneous symmetry breaking in dimension two. The system at zero temperature settles onto a classical Shimura variety, which parameterizes the pure phases of the system. The noncommutative space has an arithmetic structure provided by a rational subalgebra closely related to the modular Hecke algebra. The action of the symmetry group involves the formalism of superselection sectors and the full noncommutative system at positive temperature. It acts on values of the ground states at the rational elements via the Galois group of the modular field."

M. Planat explains:

"[This paper] is quite remarkable: it still generalizes the 1995 Bost and Connes paper to a more general Hamiltonian (proposition 1.17), which is the logarithmic determinant of 2 by 2 matrices associated to an integer lattice. Instead of the Riemann zeta funcion at temperature \beta, the partition function becomes a product of 2 Riemann zeta functions at \beta and \beta -1. This product appears also as a Mellin transform of the logarithm for the number of unrestricted partitions $p(n)$, a function I used in a recent paper, which generalizes Planck theory of radiation"

The "recent paper" was this one:

M. Planat, "Quantum 1/*f* noise in equilibrium: from Planck to Ramanujan", *Physica* A **318** (2003) 371

M. Marcolli and A. Connes, "From physics to number theory via noncommutative geometry. Part II: Renormalization, the Riemann-Hilbert correspondence, and motivic Galois theory", from *Frontiers in Number Theory, Physics, and Geometry: On Random Matrices, Zeta Functions, and Dynamical Systems* (Springer, 2006)

M. Marcolli, "Number Theory in Physics" (survey article, 07/05)

A. Connes and M. Marcolli, "A walk in the noncommutative garden" (preprint 01/06) [abstract:] "This text is written for the volume of the school/conference "Noncommutative Geometry 2005" held at IPM Tehran. It gives a survey of methods and results in noncommutative geometry, based on a discussion of significant examples of noncommutative spaces in geometry, number theory, and physics. The paper also contains an outline (the "Tehran program") of ongoing joint work with Consani on the noncommutative geometry of the adeles class space and its relation to number theoretic questions."

J. Baez, *This Week's Finds in Mathematical Physics* week 218 contains an illuminating discussion of the work of Connes, Marcolli, Haran, also framing certain issues concerning the bewildering array of zeta and *L*-functions in terms of category theory.

E. Ha and F. Paugam, "Bost-Connes-Marcolli systems for Shimura varieties" (preprint 03/05)

[abstract:] "We construct a Quantum Statistical Mechanical system (A,σ_t) analogous to the Bost-Connes-Marcolli system...in the case of Shimura varieties. Along the way, we define a new Bost-Connes system for number fields which has the "correct" symmetries and "correct" partition function. We give a formalism that applies to general Shimura data (*G,X*). The object of this series of papers is to show that these systems have phase transitions and spontaneous symmetry breaking, and to classify their KMS states, at least for low temperature." [additional background information]

Yu. Manin and M. Marcolli, "Holography principle and arithmetic of algebraic curves", *Adv. Theor. Math. Phys.* **5** (2001), no. 3, 617–650.

[abstract:] "According to the holography principle (due to G. 't Hooft, L. Susskind, J. Maldacena, *et al.*), quantum gravity and string theory on certain manifolds with boundary can be studied in terms of a conformal field theory on the boundary. Only a few mathematically exact results corroborating this exciting program are known. In this paper we interpret from this perspective several constructions which arose initially in the arithmetic geometry of algebraic curves. We show that the relation between hyperbolic geometry and Arakelov geometry at arithmetic infinity involves exactly the same geometric data as the Euclidean AdS_3 holography of black holes. Moreover, in the case of Euclidean AdS_2 holography, we present some results on bulk/boundary correspondence where the boundary is a non-commutative space."

M. Lapidus, *In Search of the Riemann Zeros* (AMS, 2008)

[from publisher's description:] "In this book, the author proposes a new approach to understand and possibly solve the Riemann Hypothesis. His reformulation builds upon earlier (joint) work on complex fractal dimensions and the vibrations of fractal strings, combined with string theory and noncommutative geometry. Accordingly, it relies on the new notion of a fractal membrane or quantized fractal string, along with the modular flow on the associated modui space of fractal membranes. Conjecturally, under the action of the modular flow, the spacetime geometries become increasingly symmetric and crystal-like, hence, arithmetic. Correspondingly, the zeros of the associated zeta functions eventually condense onto the critical line, towards which they are attracted, thereby explaining why the Riemann Hypothesis must be true.

Written with a diverse audience in mind, this unique book is suitable for graduate students, experts and nonexperts alike, with an interest in number theory, analysis, dynamical systems, arithmetic, fractal or noncommutative geometry, and mathematical and mathematical or theoretical physics."

[more relevant work by Lapidus listed here]

V. Gayral, B. Iochum and D.V. Vassilevich, "Heat kernel and number theory on NC-torus" (preprint 07/2006)

[abstract:] "The heat trace asymptotics on the noncommutative torus, where generalized Laplacians are made out of left and right regular representations, is fully determined. It turns out that this question is very sensitive to the number-theoretical aspect of the deformation parameters. The central condition we use is of a Diophantine type. More generally, the importance of number theory is made explicit on a few examples. We apply the results to the spectral action computation and revisit the UV/IR mixing phenomenon for a scalar theory. Although we find non-local counterterms in the NC ϕ^4 theory on $\T^4$, we show that this theory can be made renormalizable at least at one loop, and may be even beyond."

I. Fesenko, "Several nonstandard remarks" (preprint, 2003)

[abstract:] "This text aims to present and discuss a number of situations in analysis, geometry, number theory and mathematical physics which can profit from developing their nonstandard description or interpretation and then using it to prove standard results and/or establish standard theories."

Section 8 concerns "Nonstandard interpretations of interactions between noncommutative differential geometry and number theory."

M. Nardelli, "On the possible mathematical connections concerning noncommutative minisuperspace cosmology, noncommutative quantum cosmology in low-energy string action, noncommutative Kantowsky-Sachs quantum model, spectral action principle associated with a noncommutative space and some aspects concerning the loop quantum gravity" (preprint, 2007)